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TWENTIETH CENTURY TEXT-BOOKS

TWENTIETH CENTURY TEXT-BOOKS

A HIGH SCHOOL ALGEBRA

BY

J. W. A. YOUNG, PH.D.
ASSOCIATE PROFESSOR OF THE PEDAGOGY OF MATHEMATICS
THE UNIVERSITY OF CHICAGO

AND

LAMBERT L. JACKSON, PH.D.
FORMERLY PROFESSOR OF MATHEMATICS, STATE NORMAL
SCHOOL, BROCKPORT, NEW YORK

PART ONE

D. APPLETON AND COMPANY
NEW YORK CHICAGO

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PREFACE

This volume presents all the work required in any standard course of study for the first year of algebra. To meet maximum requirements, the topics of quadratic equations, radicals, exponents, ratio, proportion, variation, and graphs have been treated. This treatment is reviewed and extended in the authors' "A High School Algebra, Part Two," so that if, for the purposes of a shorter course, one or more of these topics are omitted from the first year's work, a class later taking up Part Two will find there a sufficient treatment of the subjects omitted.

In whatever manner the subject of geometry and algebra is alternated, the student acquires little knowledge of the metrical properties of geometry during the first year. For this reason, the authors have used in their problems only the most obvious of these properties.

Each important process of algebra is immediately applied to the solving of equations. This plan serves not only to secure the pupil's interest, but reveals to him the utility of algebra.

Great pains have been taken to supply ample practice work, and the authors have given under the most important topics, such as equations, factoring, highest common factor, fractions and exponents, a greater number of exercises than will be required by any one class. In fact, there have been included as many exercises and problems as a text-book of reasonable size will admit.

Particular attention has been given to the grading of the exercises and problems, and, for convenience in checking the results, the exercises have been so constructed that the answers are not more complex than the purpose of the exercises actually requires. The authors have followed the criterion that every principle should be exemplified with the minimum of calculation.

Among the features that contribute to the teachableness of the book are the Historical Notes. These brief sketches, describing the origin of some of the more important topics of algebra, tend to stimulate the pupil's interest, and the accompanying biographical notes and portraits of famous mathematicians serve further to humanize the subject. No attempt has been made to give a connected account of the development of algebra even in outline; these notes will serve their purpose if they create a desire to read some standard work on the history of mathematics.

Other aids which teachers will appreciate are the inductive developments, the cross references, illustrative problems, methods of testing results, careful statement of rules, topical and logical arrangement, definite classification, the frequent reviews, and the summaries of the theoretic chapters.

The authors wish to express their gratitude for the assistance rendered by those who read the manuscript and the proof sheets. For valuable constructive suggestions in preparing the manuscript, they are indebted to Mr. Allen H. Knapp, of the Central High School, Springfield, Mass., and Mr. Julius J. H. Hayn, of the Masten Park High School, Buffalo, N. Y.; while for efficient aid in reading the proofs, they owe much to Mr. Matthew R. McCann, of the English High School, Worcester, Mass., and Mr. William H. Wentworth, of the Northwestern High School, Detroit, Mich. For the portraits of famous mathematicians reproduced in this volume, they are indebted to the generosity of Professor David Eugene Smith, of Teachers College, Columbia University, New York City, who placed at their disposal his unique collection.

THE AUTHORS.

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A HIGH SCHOOL ALGEBRA

PART ONE

CHAPTER I

LITERAL NOTATION AND ITS USES

1. Numbers represented by Letters. In arithmetic, numbers are represented by means of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. But letters also may be used to stand for numbers.

For example:

p may stand for the number of *pounds* in the weight of a body;

d may stand for the number of *dollars* in a sum of money;

l may stand for the number of units in the *length* of an object, and the like.

2. The Use of Signs. The signs $+$, $-$, $=$, $\times$, $\div$, and $\sqrt{}$ have the same meaning in algebra as in arithmetic. But in algebra, multiplication is indicated also by the *absence of a sign of operation.* When a sign is needed, the *dot,* $\cdot$, is often used in preference to the symbol $\times$, which is likely to be mistaken for the letter x.

For example:

a plus b is written $a + b$, just as 3 plus 2 is written $3 + 2$.

a minus b is written $a - b$, just as 3 minus 2 is written $3 - 2$.

a divided by b is written $a \div b$ or $\frac{a}{b}$, just as 3 divided by 2 is written $3 \div 2$ or $\frac{3}{2}$.

2 times 5 is written 2×5 or $2 \cdot 5$.

a times b is written ab. 2 times a is written $2\,a$. And 2 times a plus b times c is written $2\,a + bc$.

The square root of a is written $\sqrt{a}$; the cube root of a, $\sqrt[3]{a}$; and so on.

3. The use of letters to represent numbers enables us to write statements in very brief form. This is an important feature of algebra.

For example:

1. The length of a lot diminished by $\frac{1}{5}$ of its length is 60 ft.

Using l for the number of feet in the length of the lot, this statement may be written:

l minus $\frac{1}{5}l$ is 60,
or, $l - \frac{1}{5}l = 60$.

2. A man's weight when increased by $\frac{1}{3}$ of itself is 200 lb.

Using w for the number of pounds in the man's weight, this statement may be written:

w plus $\frac{1}{3}w$ is 200,
or, $w + \frac{1}{3}w = 200$.

ORAL EXERCISES

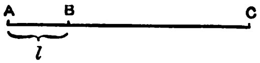
1. If l represents the number of yards in the length of a street, what stands for the length of a street 75 yd. longer?

2. If w represents the number of rods in the width of a farm, what represents the width of a farm 20 rd. narrower?

3. One bank contains d dollars and another 3 times as many. How many dollars in the second bank? How many in both banks?

4. There are n pupils in a class and the same number increased by 13 in another. How many pupils in the second class? In both classes?

5. A merchant invested s dollars and lost $\frac{1}{10}$ of this the first year. How much had he left?



6. The line BC in the figure is 3 times as long as AB. If AB is l units long, how long is BC? AC?

7. In the following statements c stands for cost, s stands for selling price, and g for gain. Read each statement in words:

1. $s - c = g$. 2. $c + g = s$. 3. $s - g = c$.

4. Algebraic Symbols. Letters and other characters used as notations in algebra are called **algebraic symbols.**

5. Algebraic Expressions. Any expression representing a number by use of algebraic symbols is called an **algebraic expression.**

For example: $3b$ and $2a - bc + d$ are algebraic expressions.

The term *literal expression* is often used to denote an algebraic expression involving letters.

6. Value of Algebraic Expressions. The number represented by an algebraic expression is called its **value**. The value of an algebraic expression is found by substituting numbers for the letters.

Thus, when $a = 3$, $2a = 2 \cdot 3$, or 6.
Similarly, when $n = 5$, $2n - 1 = 2 \cdot 5 - 1$, or 9.

ORAL EXERCISES

1. What is the value of $2n$ when n is 1? When n is 2?

2. What number is $\frac{1}{2}n$ when n is 2? When n is 6? When n is 10?

3. State the value of $2n$ when n is $\frac{1}{2}$. Also when n equals each of the following: $\frac{3}{8}$; $7\frac{1}{2}$; 10; .5; 1.5; 50.

4. State the value of $n + 1$ when n equals each of the following: 1; 2; 6; 5; $\frac{1}{2}$; .5; $8\frac{1}{2}$; 100; 0.

5. The length (l) of a box is twice its width (w). (l) is (?) w.

6. How many ounces are there in 5 lb.? In x lb.?

7. A pair of gloves costs c cents. What would the cost be if the price were raised 5 cents?

8. There were b books on a shelf and 2 were taken down. How many remained on the shelf?

9. A person is x years of age now. How old will he be a year hence? 5 years hence? How old was he 3 years ago? y years ago?

10. A merchant sold goods at 8 % above cost. If c was the number of dollars in the cost, the gain was 8 % of c, or $.08c$. What was the selling price?

11. Some goods costing x dollars were sold at a gain of 150 %. State the selling price.

WRITTEN EXERCISES

1. Write the sum of b and c, using the sign of addition.

2. Indicate the subtraction of c from b by using the sign of subtraction.

3. Write the product of a and b as it is expressed in algebra.

4. Write the product of 3 and b and c as it is expressed in algebra.

5. Indicate that a is to be divided by b by using the fractional form.

6. Indicate that the sum of a and b is to be divided by c.

7. Find the value of $2a+1$ when a equals each of the following: 4; 7; $\frac{7}{2}$; $11\frac{1}{2}$; 15; .5; 1.5; 100; 0.

Find the value of $a+b$, when a and b indicate in turn the following numbers:

8. $a=2, b=1.$ 10. $a=12, b=9.$ 12. $a=.8, b=.5.$

9. $a=14, b=6.$ 11. $a=\frac{1}{2}, b=\frac{1}{4}.$ 13. $a=1\frac{3}{4}, b=\frac{5}{8}.$

For each pair of values of a and b above, find the corresponding value of:

14. $a-b.$ 15. $ab.$ 16. $\frac{a}{b}.$ 17. $\frac{a+b}{ab}.$

18. Draw a rectangle; write b for its base and a for its altitude. Express the area of the rectangle; its perimeter (sum of its four sides).

19. A rectangular bin is a ft. long, b ft. wide, and c ft. deep. How many cubic feet does it contain?

20. What number does $100a+10b+c$ represent when $a=1$, $b=2$, $c=3$? When $a=5$, $b=4$, $c=7$?

7. Tabulation of Values. In recording corresponding values it is convenient to use a table like that adjoining.

a	$2a+1$
5	11
0	1
3	7
$8\frac{1}{2}$	18
3.5	8
⋮	

The values of a are written in the column under a, and the corresponding values of $2a+1$ are written opposite in the second column. The table records, for example, that when a is 5, $2a+1$ is 11; that is, $2\times 5+1$. Verify the other values.

WRITTEN EXERCISES

Copy the following tables and supply the numbers to fill the blanks:

1.

n	$3n$
0	( )
1	( )
4	( )
5	( )
$4\frac{1}{3}$	( )
$\frac{10}{3}$	( )

4.

n	$n-1$
1	( )
2	( )
7	( )
$6\frac{1}{2}$	( )
18	( )
25	( )

7.

v	$10v$
.1	( )
1.2	( )
1.5	( )
6.3	( )
40.4	( )
.05	( )

2.

n	$2n-3$
2	( )
$2\frac{1}{2}$	( )
3	( )
$\frac{5}{2}$	( )
1.5	( )

5.

a	$\frac{1}{2}a+1$
2	( )
12	( )
25	( )
1.8	( )
4.6	( )

8.

w	$\frac{3}{4}w$
0	( )
1	( )
24	( )
64	( )
100	( )

3.

a	b	$a+b$
4	5	( )
$6\frac{1}{2}$	$2\frac{1}{2}$	( )
1.8	9.2	( )
2.5	8.3	( )

6.

v	t	vt
$17\frac{1}{2}$	$6\frac{1}{2}$	( )
1.8	.7	( )
4	3.8	( )
12	$7\frac{1}{3}$	( )

9.

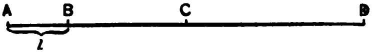
l	b	t	lbt
6	8	3	( )
4	5	6	( )
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	( )
$\frac{2}{3}$	$\frac{3}{5}$	.8	( )

REVIEW

ORAL EXERCISES

1. A man invests d dollars in real estate, and 5 times as much in government bonds. How much does he invest altogether?

2. In the figure BC is twice as long as AB, and CD is three times as long as AB. If l denotes the length of AB, what denotes the length of BC? Of CD? Of the entire line?



Find the value of $A+B$, when A and B have the following values:

3. $A=7,\ B=12.$
4. $A=\frac{2}{3},\ B=\frac{1}{6}.$
5. $A=3x,\ B=4x.$
6. $A=5y,\ B=7y.$

WRITTEN EXERCISES

1. Copy and fill the blanks:

		(1)	(2)	(3)	(4)	(5)
Given	$n=$	6	$\frac{2}{3}$	$\frac{3}{2}$	100	1.3
	$t=$	4	$\frac{1}{3}$	$\frac{5}{2}$	10	20.
Find	$3n-1=$	—	—	—	—	—
	$n+t=$	—	—	—	—	—
	$2nt=$	—	—	—	—	—

2. The length of one line is x and that of another is y. How long is the line formed by placing them end to end?

3. Line x is longer than line y. Express the difference between their lengths.

4. What number does $\frac{a}{b}$ represent when $a=29$, and $b=58$? When $a=35$, and $b=70$? When $a=127$, $b=210$?

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What symbols may stand for numbers? Sec. 1.
2. Name the different *signs of operation* used in algebra. Sec. 2.
3. In what ways may multiplication be indicated in algebra? Sec. 2.
4. What is an *algebraic expression?* Sec. 5.
5. What is meant by the *value* of an algebraic expression? Sec. 6.

HISTORICAL NOTE

If algebra appears more abstract than other school studies, if it seems less real than arithmetic, this is largely on account of its use of letters and other symbols of abbreviation. We have learned to express "any number" by a single letter, like a or x, to abbreviate the "square of a number" by a^2 or x^2, and to express operations with these numbers by signs like $+$, $-$, $\times$, $=$; all of this seems artificial and abstract, but this is one of the chief sources of algebra's power. With these symbols we can easily find in a few minutes results which the ancients sought in vain. Thus, through lack of a suitable notation, the Greeks made slow progress in studying the processes with numbers as well as those phases of geometry that require calculation, like the measuring of circles and polygons; and it was not until 300 A.D. that Diophantos, who taught in the University of Alexandria, Egypt, developed algebra into a science.

History tells little of the life of Diophantos. His birthplace, his parentage, his early education, and the steps which led to his great achievements are unknown. He died in 330 A.D., and his age at death is shown to have been 84 years by the following epitaph: "Diophantos passed one-sixth of his life in childhood, one-twelfth in youth, and one-seventh more as a bachelor. Five years after his marriage was born a son, who died four years before his father at half his father's age." It may seem strange to resort to this indirect means of telling the age of a famous man, but it was the custom of the ancients to inclose in the tombs of great men some possession or record associated with their lives; and this is one of the chief sources of early history.

Diophantos was the first to use abbreviations and symbols in algebra. On this account, and because of his simple solution of many problems then thought difficult, he has been called the Father of Algebra.

CHAPTER II

DEFINITIONS OF ELEMENTARY TERMS

8. **Product and Factors.** In algebra, as in arithmetic, the result of multiplication is called a **product**, and the numbers multiplied are called the **factors** of the product.

9. A number may be the product of different sets of factors.

10. **Literal and Numerical Factors.** Factors expressed by letters are called **literal** factors; factors expressed by numerals are called **numerical** factors.

For example:

10 is the product of the factors 2 and 5.
$2ax$ is the product of the factors 2, a, and x, in which
2 is a numerical factor, but a and x are literal factors.

In a product it is customary to put the numerical factor (if any) first, and the literal factors in alphabetical order.

11. **Commutative Law of Multiplication.** *The product is the same in whatever order the factors are taken.*

For example: $3 \cdot ab \cdot x$ gives the same product as $ab \cdot 3 \cdot x$ and the same as $abx \cdot 3$.

12. Unity is a factor of every number, but it is not ordinarily mentioned in giving lists of factors.

Thus, a set of factors of abc are 1, a, b, and c; but 1 is usually not mentioned.

ORAL EXERCISES

Name a set of factors for each of the following products:

1. 10.	3. prt.	5. $\frac{1}{2}ab$.	7. gt.
2. $3a$.	4. lr.	6. mv.	8. $\frac{1}{2}sf$.

Name three sets of factors for each of the following and state which factors are literal and which are numerical:

9. 40. **11.** $25\,ab$. **13.** prt. **15.** $75\,pq$.
10. $18\,a$. **12.** $20\,hr$. **14.** $25\,mv$. **16.** $abcd$.

13. Power and Base. A product formed by using the same number one or more times as a factor is called a **power** of the repeated factor. The repeated factor is called the **base.**

14. Exponent. When a factor is to be repeated, it is usual to write the factor only once and place a small number above and to the right to show how many times the number is to be used as a factor. The small number is called an **exponent.**

Thus, $2 \cdot 2$ is the second power (or square) of 2, and is written 2^2; $a \cdot a \cdot a$ is the third power (or cube) of a, and is written a^3. Similarly, $3\,aabbb$ is written $3\,a^2b^3$.

15. A number without an exponent is understood to have the exponent 1, since the number is used once as a factor.

Thus, $5 = 5^1$; $a = a^1$; $7\,xyz^3 = 7^1x^1y^1z^3$.

16. The factors of numbers can be conveniently grouped by the use of exponents.

For example:

$$12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3. \qquad 144 = 12 \cdot 12 = 12^2.$$
$$225 = 3 \cdot 3 \cdot 5 \cdot 5 = 3^2 \cdot 5^2. \qquad 600 = 2^3 \cdot 3 \cdot 5^2.$$

17. Prime Numbers. As in arithmetic, so in algebra, an integer whose only integral factors are itself and unity is called a **prime number.**

18. Prime Factors. Prime numbers occurring as factors are called **prime factors.**

A number has only one set of *prime* factors.

ORAL EXERCISES

Name the exponents and tell what each means:

1. $2\,a^2x^2$. 2. $3\,x^3$. 3. $5\,xy^3$. 4. a^4b^4. 5. $2\,a^5y$.

WRITTEN EXERCISES

Regarding each letter as prime, indicate the prime factors of:

1. 18. 3. 96. 5. 640. 7. 360. 9. 1225.
2. $75a$. 4. $12x^2$ 6. $24ab$. 8. $17a^2b$. 10. $38x^3y$.

19. Coefficient. Any factor in a product is called the **coefficient** of the rest of the product.

Thus, in the product $3axy$, 3 is the coefficient of axy, $3a$ is the coefficient of xy, $3ax$ is the coefficient of y, $3ay$ is the coefficient of x, and the like.

20. Numerical Coefficient. A coefficient expressed in numerals is called a **numerical coefficient.**

Thus, 3 is the numerical coefficient in $3x$, and $\frac{1}{2}$ is the numerical coefficient in $\frac{1}{2}xy$.

The term "coefficient," used with no other indication, means the numerical coefficient.

21. In any product whose numerical coefficient is not expressed, the coefficient 1 is understood.

Thus, ab, abx, bc^2y, are the same as $1ab$, $1abx$, $1bc^2y$; and the numerical coefficient in each is 1.

ORAL EXERCISES

1. In $6ab$ name the coefficient of ab. The coefficient of b.
2. In $\frac{2}{3}axy$ name the coefficient of xy. Of axy. Of y.

Name the numerical coefficient in each of the following:

3.	4.	5.	6.	7.	8.	9.	10.	11.
$2x$.	$3y$.	$\frac{2}{3}ax$.	by.	$.5cz$.	$\frac{4}{3}my$.	$\frac{1}{2}gt^2$.	xyz.	$\frac{1}{2}mr^2$.

22. Order of Operations. *In an expression containing a series of operations, multiplications and divisions are to be performed before additions and subtractions, unless otherwise indicated.*

Thus, $4 + 5 \cdot 3$ means $4 + 15$, or 19.
Similarly, $3 + 8 \div 2 - 5 = 3+4-5 = 7 - 5 = 2$.

ORAL EXERCISES

Perform the operations indicated:

1. $5 \cdot 4 - 3$.
2. $9 + 2 \cdot 6$.
3. $14 - 8 \div 4$.
4. $15 - 12 \div 4$.
5. $40 - 34 \div 2$.
6. $52 \div 26 - 1$.
7. $10 \cdot 3 - 3 \cdot 8$.
8. $3 \cdot 9 - 15 \div 5$.
9. $18 \div 2 + 3 \cdot 6$.
10. $3 + 4 \cdot 5 - 13$.
11. $a + 2a \div a - a$.
12. $b \cdot a + a \cdot b$.

23. The Use of the Parenthesis. In a series of operations the parenthesis may be used to indicate that certain additions and subtractions are to be performed first.

For example:

$6 + 4 \cdot 3$ means add 6 to 4 times 3, obtaining 18. But $(6 + 4) \cdot 3$ means add 6 and 4 and multiply the result by 3, obtaining 30. In other words, what is in the parenthesis is to be treated as a single number.

$(4 + 16) \div 2$ means $20 \div 2$, or 10, and not $4 + \frac{16}{2}$, or 12.

$8 - (12 - 7)$ means that 7 is first to be subtracted from 12 and then the result subtracted from 8. That is, $8 - (12 - 7) = 8 - 5 = 3$.

$14a - (7a + 3a) = 14a - 10a = 4a$.

24. When a number symbol is placed before or after a parenthesis with no intervening sign, multiplication is indicated.

For example:

$2a(c + 2c)$ means $2a \cdot 3c$, or $6ac$.
$(4x + 5x)4x$ means $9x \cdot 4x$ or $36x^2$.
$(a + 4a)(8b - 5b)$ means $5a \cdot 3b$, or $15ab$.

$3(b + 2c) - c$ means $3b + 6c - c$, the multiplication by 3 being performed before c is subtracted.

$5(100 - x) + 25$ means $500 - 5x + 25$, or $525 - 5x$, the multiplication by 5 being performed before 25 is added.

ORAL EXERCISES

Perform the operations indicated:

1. $(15 - 6) \div 3$.
2. $7(25 + 5)$.
3. $(18 - 12) \div 6$.
4. $5(6 + 5 - 9)$.
5. $(2 + 5) \cdot (5 - 3)$.
6. $(24 + 6) \div (8 - 3)$.
7. $(2a + a) \div 3$.
8. $(2a + 3a)b$.
9. $(2a + 3a) \div (3b + 2b)$.
10. $a - (b + 2b - 3b)$.

11. $ab(3c-2c)+d$.	15. $19-(4+7)$.
12. $m(m+5m)-2m$.	16. $8a-(7a-3a)$.
13. $(2a+a)\cdot(3c-c)$.	17. $43x+(28x-8x)$.
14. $x(5x-2x)-y(y+4y)$.	18. $86y-(10y-4y)+10y$.

25. Symbols of Grouping. The parenthesis is used to indicate that the number symbols grouped within it are to be taken as a single number. Other symbols of grouping are the brace, $\{\ \}$, the bracket, $[\ ]$, the bar, $\overline{\quad}$; these have the same meaning as the parenthesis. The bar of the fraction may also be a symbol of grouping.

Thus, in $\frac{a+b}{c+d}$, the bar groups $a+b$ into one number, and $c+d$ into one number. The fraction means $(a+b)\div(c+d)$.

26. Monomials. A **monomial** is an algebraic expression within which no operation of addition or subtraction is indicated, unless within a symbol of grouping.

Thus, a, ab, $a\div 2b$, $\frac{5ab}{c^2}$, $7(a+b)$, $\frac{2c-5}{x}$, are monomials.

27. Polynomials. An algebraic expression consisting of two or more monomials connected by the sign + or − is called a **polynomial.** The monomials are called the **terms** of the polynomial.

Thus, $a+5b+c+\frac{d}{2}$ is a polynomial whose terms are a, $5b$, c, and $\frac{d}{2}$.

28. Binomials. A polynomial of two terms is called a **binomial.**

Thus, $a+b$, b^2-c, $xy+m$, $3b^2-a$, $\frac{cd}{2}-\frac{z}{w^2}$, are binomials.

29. Trinomials. A polynomial of three terms is called a **trinomial.**

Thus, $a+b+c$, $a+2b-c$, $\frac{x}{y}-ab+3c$, are trinomials.

30. Compound Terms. Expressions are sometimes grouped into **compound terms.**

Thus, $3a-2b+c+d$ may be grouped into the trinomial $3a-2b+(c+d)$, in which $(c+d)$ is a compound term.

ORAL EXERCISES

Name the monomials in the following list; the binomials; the trinomials:

1. $a + b - c.$
2. $4x^2 + 7.$
3. $a + b + c + d.$
4. $\frac{1}{2}gt^2.$
5. $gt^2 + a.$
6. $mv^2.$
7. $a + 5b - c.$
8. $x - y + z + w.$
9. $5x^2 + ay^3.$
10. $2hr.$
11. $prt.$
12. $2g - 5 + x^2.$
13. What is the coefficient of t^2 in Exercise 4?
14. What is the coefficient of v^2 in Exercise 6?
15. What is the numerical coefficient in Exercise 4?
16. Name the numerical coefficients in Exercise 7.
17. Name the coefficient of h in Exercise 10.
18. Read the numerical coefficients in Exercise 8.

WRITTEN EXERCISES

1. Write three monomials.
2. Write three binomials. Three trinomials.

Rewrite these expressions, using exponents where possible:

3. $a + bb.$
4. $2aa + b.$
5. $aa - bbb.$
6. $ccc - bb.$
7. $3aayyy.$
8. $2 \cdot 2 \cdot 2bbb.$
9. $15mvvq.$
10. $16xxy - cd.$
11. $100aabb - sss.$

Using $a = 1$, $b = 2$, and $c = 3$, find the value of each of the following polynomials:

12. $5a + 9b.$
13. $10a - 5b.$
14. $2a + b - c.$
15. $3a + 15b.$
16. $2a + 3b + 3c.$
17. $.9a + .3b - .1c.$
18. $9b + 2a - c.$
19. $2a + 3b - 2c.$
20. $3a + 7b + 11c.$
21. $3a - 7b + 11c.$
22. $61b - 2c - 20a.$
23. $\frac{1}{2}a + \frac{2}{3}c - \frac{1}{6}b.$
24. $a^2 + b^2.$
25. $b^3 - a^2.$
26. $ac^2 + 3b^3.$
27. $7ab^2 - c^3.$
28. $c^3 - b^3.$
29. $c^4 - 1 + 5b^3.$

31. Uses of Monomials. Monomials have various uses. For example:

1. *They are used as formulas in business arithmetic.*

Thus:

br is often used as a short way of stating *base times rate* in percentage.

prt is often used as a short way of stating *principal times rate times time* in interest.

When any particular value of r is substituted in the above formulas it must be expressed decimally. Thus, if r is 5%, it must be used as .05, or $\frac{5}{100}$.

2. *They are used as formulas of measurement.*

Thus:

ab is often used as a short way of stating *altitude times base* in finding areas.

abc is a short way of stating *length times breadth times thickness*, in finding volumes of rectangular solids, where a, b, and c are the edges.

π (read "pī") is used to denote *the number by which the length of the diameter of a circle must be multiplied to produce the circumference.*

The value of π is approximately 3.1416. Letting $c =$ circumference, and $d =$ diameter, we have $c = \pi d = 3.1416\,d$.

The word circumference, as used above, means the distance around the circle or the length of the curve. The conception of a circle as a curve is the one used in advanced mathematics, in other sciences, and in common parlance.

3. *They are used to express laws of physics.*

Thus, *vt* is often used as a short way of stating *product of velocity and time* in finding distance.

WRITTEN EXERCISES

1. Percentage $= br$. Find the percentage when $b = 400$ and $r = 30\,\%$. In substituting r use $\dfrac{30}{100}$.

2. Rate $= \dfrac{p}{b}$. Find the rate when $p = 15$ and $b = 750$.

3. Principal $= \dfrac{i}{rt}$. Find the principal when $i = \$500, r = 5\,\%$, and $t = 10$ yr.

4. Interest $= prt$. Find the interest when $p = \$100$, $r = 5\,\%$, and $t = 5$ yr.

5. Discount $= lr$. Find the discount when $l = \$820$ and $r = 12\frac{1}{2}\%$.

6. Rate of discount $= \frac{d}{l}$. Find the rate when $d = \$35$ and $l = \$875$.

7. The area of a rectangle $= ab$. Find the area when $a =$ 20 in. and $b = 17\frac{1}{2}$ in.

8. What is the area of a rectangle when $a = 4x$ inches and $b = 3x$ inches?

9. Copy the following table and fill out the blanks, using the value of π mentioned in Sec. 31. Answer from your table:

(*a*) What is the circumference of a circle whose diameter (d) is 3 in.?

(*b*) Of one whose diameter is 1.5 in.?

(*c*) Of one whose diameter is $\frac{1}{2}$ in.? 10 ft.?

d	πd
2 in.	6.2832 in.
3 in.	( )
$\frac{1}{2}$ in.	( )
1.5 in.	( )
10 ft.	( )

10. The distance (d) traveled by a body in time (t) moving with velocity (v) is vt. Copy the following table and fill out the blanks. Answer from your table:

(*a*) How far will a train moving 30 ft. per second go in 2 sec.?

(*b*) How far will a train moving 50 mi. per hour travel in 2 hr.?

(*c*) How far will a bullet traveling 400 ft. per second go in 5.5 sec.?

v	t	$vt = d$
30	2	60
50	2	( )
36	$1\frac{1}{4}$	( )
400	5.5	( )
150	17	( )

11. The number of square units in the area of a triangle is $\frac{1}{2}$ of the product of the numbers of linear units in its altitude (a) and base (b). Copy this table and fill out the blanks.

Answer these questions from your table:

(*a*) What is the area of a triangle of altitude 10.5 in. and base 8 in.?

(*b*) What is the area of a triangle of altitude 70 ft. and base 9 ft.?

(*c*) What is the area of a triangle of altitude 3.3 yd. and base 5 yd.?

a	b	$\frac{1}{2}ab$
6	5	15
10.5	8	( )
70	9	( )
3.3	5	( )
9	1.7	( )

32. Uses of Polynomials. Polynomials, like monomials, have various uses as formulas.

For example:

$l - lr$ may stand for *list price – discount*, or net price.

$c + rc$ may stand for *cost + rate of gain times the cost*, or the selling price.

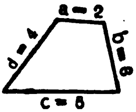
$2a + 2b$ may stand for *the perimeter of a rectangle of sides a and b.*

$2ab + 2ac + 2bc$ may stand for *the surface of a rectangular solid of edges a, b, c.*

ORAL EXERCISES

1. What is the value of $l - rl$ when $l = \$100$ and $r = 5\%$? What is the net price of goods listed at \$100 and bought at a discount of 5%?

2. What is the value of $c + cr$, when $c = \$200$ and $r = 10\%$? What is the selling price of goods which cost \$200 and are sold at a gain of 10%?



3. When $a = 3$, $b = 4$, $c = 5$, $d = 6$, what is the value of $a + b + c + d$?

4. What is the value of $a + b + c + d$ in the figure?

5. What is the value of $2a + 2b$ when $a = 3$, $b = 5$? What is the perimeter of a rectangle whose sides are 3 yd. and 5 yd.?

WRITTEN EXERCISES

1. When $a = 10$, $b = 15$, $c = 24$, find the value of $2ab + 2ac + 2bc$. What is the area of the surface of a rectangular prism whose edges are 20 in., 15 in., 50 in.?

2. Find the value of $2ab + 2bc + 2ac$, when $a = 20$, $b = 25$, $c = 50$.

3. Find the value of $a^2 + b^2$, when $a = 13$, $b = 210$; also when $a = 75$, $b = 100$.

4. Find the value of $a^2 + b^2 + c^2$, when $a = 35$, $b = 20$, $c = 65$; also when $a = 100$, $b = 75$, $c = 150$.

5. Find the value of $ut + \frac{1}{2}at^2$, when $u = 1500$, $a = 200$, $t = 10$.

33. Degree of a Monomial. The degree of a monomial is the sum of the exponents of its literal factors.

Thus : a^2 is of the second degree.
$3\,ab$ is of the second degree.
$2\,a^3b$ is of the fourth degree.

But the degree is often expressed with respect to some letter or letters.

Thus, $3\,ax^2y^3$ is of the first degree with respect to a, of the second degree with respect to x, of the third degree with respect to y, and of the fifth degree with respect to x and y.

34. Degree of a Polynomial. The degree of a polynomial is that of its term of highest degree; its degree with respect to a letter is the highest degree of that letter in the polynomial.

Thus, $a^3x^2 - 5\,by + xy^4z$ is of the sixth degree; it is of the third degree in a, the first in b and in z, the second in x, the fourth in y, and the sixth in x, y, and z.

NOTE. It is not necessary in elementary algebra to define the degree of expressions containing radicals or fractions.

ORAL EXERCISES

State the degree of each of the following monomials:

1. a^2b.
2. ax.
3. $2\,ax$.
4. $3\,a^2x$.
5. a^3b.
6. $4\,a^2b^2$.
7. $5\,mn^2$.
8. $9\,xyz$.
9. $9\,x^2bz$.
10. x^2yz^2.
11. $\frac{1}{2}\,mv$.
12. $\frac{1}{2}\,gt^2$.

13. State the degree of the expressions in Exercises 1–6 with respect to a; in Exercises 8–10 with respect to x.

14. State the degree of the expressions in Exercises 4–9 with respect to each letter involved.

State the degree of each polynomial; also its degree with respect to each letter:

15. $ab^2 + b$.
16. $a^2b + a^2c + d$.
17. $3\,x^2 + 2\,x + 1$.
18. $abc + c^3 + bcd$.
19. $3\,ax + 3\,x^2y + y^3$.
20. $\frac{1}{2}\,m^2 + n^2 + 3\,pq$.

REVIEW

ORAL EXERCISES

State the product of each set of factors:

1. $8, 6, a$. **2.** $b, x, 3$. **3.** $a, y, 3, 4$. **4.** $2, x, a, x$.

Name three sets of factors for each of the following:

5. $24\,mx$. **6.** ax^2y. **7.** $\frac{1}{2}gt^2$. **8.** $21\,abc$.

In each of the following name (1) the coefficient of x; (2) the numerical coefficient:

9. $4\,ax$. **10.** $12\,bx$. **11.** acx. **12.** $5\,mxy$. **13.** $12\,c^2x$.

State the value of:

14. 2^4. **15.** 7^2. **16.** 5^3. **17.** $3^2 \cdot 2^4$. **18.** $2^3 \cdot 5^2$.

From the following list select by number the binomials; the trinomials; the monomials:

19. ax^2.
20. $a - x$.
21. $2\,x^3 - 5\,a$.
22. $6\,a + 7\,xy$.
23. $3\,ay^2 - 4\,x + 1$.
24. $2\,ab + 7\,x^3 - 5\,x^2$.
25. $a^2 - 2\,ab + b^2$.
26. $x^3 + 3\,x^2 + 3\,x + 1$.

27. State the degree of each expression in Exercises 19–26 with respect to x. With respect to a.

28. The sides of a triangle are $3\,a$, $2\,b$, $5\,c$. What is its perimeter? What kind of polynomial is this?

29. What is the value of $c + cr$ when $c = \$500$ and $r = 20\,\%$?

30. What is the value of $x^2 + 5\,a$ when $x = 3$ and $a = 2$?

State the results of the indicated operations:

31. $39 \div 13 + 1$.
32. $8 + 5 \cdot 2$.
33. $7 \cdot 8 - 6$.
34. $39 \div (13 + 1)$.
35. $(8 + 5)2$.
36. $7(8 - 6)$.
37. $8 - 5 + 2$.
38. $7\,a - 3\,a + 2\,a$.
39. $8\,x + 5\,x - x$.
40. $8 - (5 + 2)$.
41. $7\,a - (3\,a + 2\,a)$.
42. $8\,x + (5\,x - x)$.

WRITTEN EXERCISES

Indicate the prime factors of the following, using exponents where possible:

1. 88. **2.** 144. **3.** 200. **4.** 525.

Factor so that one factor of each is a power of 10:

5. 1000. **6.** 900. **7.** 23,000. **8.** 3,000,000.

Find the value of each of the following if $a = 2$, $b = 3$, $x = 1$:

9. $4ax + 1$. **10.** $\frac{3ax^2 - 1}{2b}$. **11.** $\frac{b^4 - a^4}{x + 1}$. **12.** $\frac{a^4 + 4}{b + 2x}$.

13. The area of a triangle is $\frac{1}{2}ab$. What is the area of a triangle in which $a = 3n$ feet and $b = 14n$ feet?

14. Find the value of $a^2 + 2ab + b^2$ when $a = 7$, $b = 3$.

15. Find the value of $a^3 - 3a^2x + 3ax^2$ when $a = 7$, $x = 2$.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define *product.* Also *factor.* Sec. 8.

2. What is a *numerical* factor? A *literal* factor? Sec. 10.

3. Explain the *order* of the factors in a product. Sec. 10.

4. State the *Commutative Law of Multiplication.* Sec. 11.

5. Define *exponent;* also *base;* also *power.* Secs. 13, 14.

6. Define *coefficient;* also *numerical coefficient.* Secs. 19, 20.

7. Name the usual symbols of grouping. Secs. 23, 25.

8. What process is indicated by a number standing next to a parenthesis with no intervening sign? Sec. 24.

9. Define *monomial;* also *polynomial.* Secs. 26, 27.

10. Define *binomial;* also *trinomial.* Secs. 28, 29.

11. What is a *compound term?* Sec. 30.

12. How is the *degree of a monomial* found? Also the degree with respect to a given letter? Sec. 33.

13. What is the *degree of a polynomial?* Sec. 34.

HISTORICAL NOTE

To understand how the symbols of algebra came to be what they are, we may again refer to Diophantos, for in his work, *Arithmetica*, he made the first approach to algebraic notation by using certain signs and abbreviations. Since these characters are formed from Greek letters and words, it would be confusing to illustrate more than a few of them, and we will confine ourselves to showing how Diophantos would have written the polynomial, $3x^2 + x - 2$. His coefficients were placed after the literal factors; hence, $3x^2$ would be $x^2 3$, and x would be $x 1$. We must also remember that the coefficients were expressed by the Greek numerals: $\bar{\alpha}$ for 1, $\bar{\beta}$ for 2, $\bar{\gamma}$ for 3, and so on. For x he used the symbol ς, probably a contraction of the first two letters of the Greek word meaning "unknown number." Diophantos did not know of exponents, but for x^2 he used $\delta^{\hat{\upsilon}}$, an abbreviation of the word meaning "square." His sign for minus was $\pitchfork$, derived from the letters of the word meaning "minus." He had no sign for plus, but placed numbers next to each other to indicate addition. Thus, $3x^2 + x - 2$, if written by Diophantos,
would be arranged thus, $x^2\ 3\ x\ 1 - 2$
and look like $\delta^{\hat{\upsilon}}\ \bar{\gamma}\ \varsigma\ \bar{\alpha}\ \pitchfork\ \bar{\beta}$

The notation of algebra, as we now use it, was developed about the sixteenth century. By that time the processes came to have signs to indicate them; for example, addition and subtraction were denoted by $\bar{p}$, $\bar{m}$, probably abbreviations for the words "plus" and "minus," and these were finally supplanted by + and −. The latter forms may have resulted from the rapid writing of the letters p and m, but a more likely origin is found in the commercial arithmetic of that time. In recording weights of merchandise as marked in the warerooms, the sign + was used to denote overweight, and the sign − to denote short weight. Thus, if a bale supposed to weigh 100 pounds weighed 103 pounds, it was recorded as 100 + 3, or if it weighed 97 pounds, it was recorded as 100 − 3. From this practice the signs + and − probably came to be generally used as signs of addition and subtraction. The sign × was first used by the English mathematician Oughtred in 1631, and the simpler sign, · , for multiplication was used by the famous French mathematician, René Descartes. For the square of a he used aa, which later became a^2. The symbol for square root became ℞, the initial letter of the Latin word *radix* ("root"); this was afterward changed to $\surd$ by the German algebraist, Stifel (1554). (The abbreviation ℞, now used in physicians' prescriptions, denotes similarly the first letter of the Latin word *recipe* — "take.") The sign of equality (=) was first used by Robert Recorde, an Englishman, 1542. His explanation of this sign was that "Noe. 2. thynges can be moare equalle" than these parallel lines.

CHAPTER III

THE EQUATION

35. PREPARATORY.

1. What weight, w, will balance the package of rice in the figure? What must w be to balance two such packages?

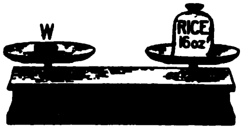


FIG. 1.

2. If half of the rice be taken from the package, what must w be?

3. What must w be to balance a 16-ounce package and an 8-ounce package?

4. What must w be to balance the pans in each of the following figures?

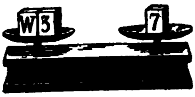


FIG. 2.

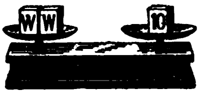


FIG. 3.

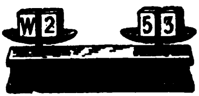


FIG. 4.

5. In Fig. 2 the fact that the weights balance is expressed by $w + 3 = 7$. Express the condition that the weights balance in Fig. 3. In Fig. 4.

36. The Equation. If two expressions represent the same number, their equality may be indicated by the sign $=$; such a statement of equality is called an **equation.**

Thus, $w + 3 = 7$, $2w = 8$, $y = 3 + 4$, and $2x + 1 = 9$ are equations.

37. Members of an Equation. The two expressions connected by the sign of equality are called the **members** of the equation.

Thus, in $w + 3 = 7$, $w + 3$ and 7 are members; they are called respectively the *left* member and the *right* member, or, also, the *first* member and the *second* member.

38. Identical Equation. An equation that either involves no letters, or that is true for any values whatever that may be given to the letters involved, is called an **identical equation** or, briefly, an **identity.**

Thus, $3 + 4 = 7$, $a + b = b + a$, $5x = 8x - 3x$, are identities

39. Conditional Equation. An equation that is true only on condition that the letter or letters involved have particular values is called a **conditional equation** or, briefly, an **equation.**

Thus, $w + 3 = 7$ is an equation that is true only on condition that w represents 4.

The equation is an abbreviated sentence. The identical equation is a declarative sentence and states that the two members are necessarily the same — differing, at most, in form. The conditional equation is an interrogative sentence, and asks what numbers must be used in place of the letters in order that the two members may be equal.

ORAL EXERCISES

1. What must be added to 3 to make 10? State the number for which the question mark stands in $3 + ? = 10$.

2. State the number denoted by the question mark in each case: $5 + ? = 12$; $? + 15 = 25$; $60 = 45 + ?$

3. State the number for which x stands in each of the following: $5 + x = 12$; $x + 15 = 25$; $60 = 45 + x$; $25 + x = 40$.

State the number denoted by the question mark in each case:

4. 2 times $? = 12$.
5. 5 times $? = 30$.
6. $\frac{1}{2}$ of $? = 8$.
7. $\frac{3}{2}$ of $? = 12$.
8. 2 times $? + 1 = 9$.
9. 3 times $? + 5 = 11$.

10. State the number represented by t in each case: $2t = 8$; $4t = 32$; $\frac{1}{2}t = 10$; $\frac{3}{2}t = 30$; $2t + 1 = 11$.

11. A certain number less 4 is 20. What is the number?

12. A certain number less 5 is 15. What is the number?

13. x less 4 is 20. What is x? $2x$ less 5 is 25. What is $2x$?

14. In $3x + 5 = 17$, what is $3x$? What is x?

15. In $4p + 2 = 10$, what is $4p$? What is p?

40. In equations we need to express relations between numbers by algebraic symbols.

For example, a number that is greater than n by 4 may be written $n+4$.

Or, if the sum of two numbers is 11 and one of them is n, the other is expressed by $11-n$.

Or, if the difference of two numbers is 15 and the larger of them is x, the other is expressed by $x-15$.

WRITTEN EXERCISES

Represent a number:

1. Greater than n by 7.
2. Greater than n by 3.
3. Less than n by 5.
4. Less than n by a.
5. Greater than $2n$ by b.
6. $2b$ less than 5 times n.
7. 3 times n less one.
8. 3 greater than $a-b$.
9. a greater than $b+c$.
10. a less than $2c-b$.
11. 6 greater than 4 times x.
12. $2a-1$ greater than $b+2$.

Write the other part if:

13. One part of 12 is 7.
14. One part of x is 3.
15. One part of 10 is y.
16. One part of n is a.
17. One part of a is n.
18. One part of $a-b$ is x.

19. The difference between two numbers is 8. The smaller of the two is 12. Write the larger.

20. The difference between two numbers is d. The smaller of them is a. Write the larger.

21. The difference between two numbers is 10. The larger is 25. Write the smaller.

22. The difference between two numbers is 21. The larger is y. Write the smaller.

23. The difference between two numbers is d. The larger is $2c$. Write the smaller.

24. $4n$ plus a number is 26. Write the number in symbols.

25. The sum of three numbers is 45. One of them is 6, another is a. Write the third.

26. The sum of three numbers is s. One of them is a, another $2b$. Write the third

Express by how much:

27. 18 exceeds a.

28. 17 exceeds $2c$.

29. 24 exceeds $a-b$.

30. 36 exceeds $2a-c$.

31. 12 is less than n.

32. 15 is less than $2a$.

33. 44 is less than $b+c$.

34. $35+a$ exceeds $b+c$.

If A is n years old, express his age:

35. 5 years ago.

36. x years ago.

37. 5 years hence.

38. x years hence.

39. $a-3$ years ago.

40. $2a-5$ years hence.

41. The translation of algebraic expressions into words assists in interpreting equations.

EXAMPLES

1. $n-3=7$ is translated "a number less three is seven."

2. $2x-1=5$ is translated "twice a number less one is five."

3. $3x-2=5x-8$ is translated "three times a number less two equals five times the number less eight."

4. $2n-m=4$ is translated "twice a number less a second number is four."

5. $A=\frac{1}{2}ab$, referring to a triangle, is translated "the area of a triangle is one half the product of the base and altitude."

6. $V=\pi r^2a$, referring to a circular cylinder, is translated "the volume of a cylinder is *pi* times the square of the radius times the altitude."

ORAL EXERCISES

Translate into words:

1. $n+5=8$.

2. $x-1=10$.

3. $3x-2=12$.

4. $5x-1=x+4$.

5. $m+n=17$.

6. $m-n=1$.

7. $2x+3y=15$.

8. $n, n+1, n+2$.

9. $2n, 2n+2, 2n+4$.

10. $n, n-1, n-2$.

11. $2n, 2n-1, 2n-3$.

12. $2x-1=2b+1$.

13. $3p+6=4q-6$.

14. $pq+1=2pq-10$.

15. Taking A to mean the area of a rectangle, read $A=ab$.

16. Taking V to mean the volume of a rectangular solid, of dimensions a, b, and c, read $V=abc$.

17. Taking A to mean the area of a circle, read $A = \pi r^2$.

18. Taking C to mean the circumference of a circle, read $C = 2\,\pi r$.

19. Taking I to mean simple interest, read $I = prt$. Also read $t = \frac{I}{pr}$.

42. Substitution. A number symbol put in place of another is said to be **substituted** for it.

For example:

$5a + 2$ becomes $5 \cdot 3 + 2$ when 3 is substituted for a; and $ax + 7$ becomes $ab + 7$ when b is substituted for x.

43. Unknown. A number symbol whose value is not known is called an **unknown number**, or simply an **unknown.**

44. Satisfying an Equation. If an equation becomes an identity when certain numbers are substituted for the unknowns, the numbers substituted are said to **satisfy** the equation.

Thus, 5 is said to *satisfy* the equation $3\,x = 15$, because $3 \times 5 = 15$. The equation is not satisfied by any other number, because 3 times any other number is not 15. Also, 7 and 5 are said to satisfy the equation $3\,x + 2\,y = 31$, because $3 \cdot 7 + 2 \cdot 5 = 31$.

45. Root of an Equation. A number that satisfies an equation is called a **root** of the equation.

46. Solving Equations. To **solve** equations is to find their roots.

ORAL EXERCISES

What number satisfies each of the following equations?

1. $3\,x = 6$.
2. $9\,x = 18$.
3. $7\,x = 35$.
4. $4\,x = 32$.
5. $5\,x + 2 = 22$.
6. $8\,x + 12 = 20$.
7. $7\,y + 5 = 40$.
8. $2\,y + 1 = 3$.
9. $30 - 6 = 4\,y$.
10. $4\,w + 2 = 10$.
11. $4\,w + 6 = 46$.
12. $\frac{1}{2}z + 3 = 9$.
13. $2\,n = 90$.
14. $2\,y + 7 = 13$.
15. $\frac{1}{8}w + 3 = 2$.
16. $2\,n = 4800$.
17. $2\,n + 1 = 27$.
18. $2\,n + 1 = 625$.

Solve the equations:

19. $4x = 20$.

20. $3y + 4 = 25$.

21. $4t + 1 = 27$.

22. $2r + 5 = 13$.

23. $4v + 1 = 9$.

24. $6u = 14 - 2$.

25. $8 + 2 = 5s$.

26. $7x + 7 = 28$.

27. $14 = 6x + 2$.

28. $20 - 4 = 4y$.

29. $9r = 360$.

30. $17u = 3400$.

31. $20p = 50 - 10$.

32. $60 + 15 = 25x$.

33. $18y = 360$.

47. PREPARATORY.

If two weights are in balance, and if the following changes are made in one weight, what change, in each case, must be made in the other to preserve the balance?

1. Two ounces added. **2.** Two ounces taken away.

3. The number of ounces in one weight made three times as great.

4. The number of ounces in one weight made $\frac{1}{4}$ as great.

48. Properties used in Solving Equations. The preceding exercises suggest the following properties:

1. *If the same number is added to equal numbers, the results are equal.*

2. *If the same number is subtracted from equal numbers, the results are equal.*

3. *If equal numbers are multiplied by the same number, the results are equal.*

4. *If equal numbers are divided by the same number (not zero), the results are equal.*

NOTE. The reason for excluding zero as a divisor is explained in Chapter IX.

49. The following examples show how these properties are used in solving equations:

EXAMPLES

1. Solve: $3x + 5 = 23$. *(1)*

Subtracting 5 from both members, $3x = 18$. *(2)*

Dividing both members of *(2)* by 3, $x = 6$. *(3)*

TEST. 6 *satisfies* $3x + 5 = 23$, because $3 \cdot 6 + 5 = 23$.

2. Solve: $p + 2 + \frac{1}{3}p = \frac{2}{3}p + 6.$ (*1*)

Subtracting $\frac{2}{3}p$ from both members, $\frac{2}{3}p + 2 = 6.$ (*2*)

Subtracting 2 from both members of (*2*), $\frac{2}{3}p = 4.$ (*3*)

Dividing both members of (*3*) by $\frac{2}{3}$, $p = 6.$ (*4*)

Test. 6 *satisfies* (*1*), because $6 + 2 + \frac{1}{3} \cdot 6 = \frac{2}{3} \cdot 6 + 6.$

$10 = 10.$

50. Testing. The correctness of the work of solving an equation should be tested by substituting the result in the given equation. If the members become identical, the number substituted is a root of the equation.

WRITTEN EXERCISES

Solve and test:

1. $4x + 1 = 7.$
2. $3x + 1 = 10.$
3. $5x = x + 16.$
4. $2x + 7 = 27.$
5. $16t + 5 = 37.$
6. $14x = 25 + 9x.$
7. $48 = 8y + 16.$
8. $12t + 13 = 49.$
9. $3s + 2 = 19.$
10. $z + 4z = 85.$
11. $15s + 2 = 12.$
12. $\frac{1}{2}x + 2 = \frac{7}{8}.$
13. $6y + 2 = 20.$
14. $8z + 2 = 42.$
15. $72 = 12x.$
16. $6x = 9 + 3x.$
17. $\frac{3}{2}x = 25 + \frac{1}{2}x.$
18. $11y + 1 = 9y + 3.$
19. $3x + 2x = 4x + 16.$
20. $11m + 3 = 2m + 9 + 2m.$
21. $8p + 5 = 2p + 14 + 3p.$
22. $12z + 28 = 7z + 53.$
23. $40 + 3z = 58 + z.$
24. $\frac{2}{3}y + 45 = \frac{1}{3}y + 55.$
25. $\frac{4}{5}y + 31 = \frac{2}{5}y + 41.$
26. $\frac{5}{7}x + 20 = 32 + \frac{3}{7}x.$

51. Use of the Equation in Solving Problems. Equations may be used in solving problems.

EXAMPLES

1. If a certain number is doubled and 16 is added to the product, the result is 46. What is the number?

Solution. 1. Let n be the number.

2. Then $2n + 16$ is double the number plus 16.
3. But 46 is given as double the number plus 16.
4. Therefore, $2n + 16 = 46.$
5. Therefore, $2n = 30$, and $n = 15.$ Why?

Test. 15 doubled makes 30, and 30 plus 16 is 46.

2. A salesman sold twice as many articles on Friday as on Thursday, and 5 more on Saturday than on Friday; on Saturday he sold 15. How many did he sell on Thursday?

SOLUTION. 1. Let x be the number that he sold on Thursday.
2. What does $2x$ represent? $2x + 5$?
3. State two expressions, each of which is the number sold on Saturday.
4. Since $2x + 5 = 15$, $x = 5$.

TEST. $2 \cdot 5 + 5 = 15$.

Any letter may be used to represent the unknown, as y for the number of years, d for the number of dollars, r for rate, or p for the number of pounds pressure as in physics, but in algebra x is most frequently used.

52. Finding the Equation. In each solution above, step 4 contains the statement of the problem in the form of an equation. *This statement is reached by finding two expressions for the same number and using them as the members of an equation.*

53. Sign of Deduction. Instead of the word "hence," or "therefore," the sign $\therefore$ is often used. It is called the **sign of deduction.**

Thus, $\therefore 2n + 16 = 46$ is read, "Therefore, $2n + 16 = 46$."

WRITTEN EXERCISES

Write the solution of each problem in steps as shown above:

1. A house and lot are worth $4800, and the house is worth 7 times as much as the lot. Find the value of each.

2. Lucy thought of a number, doubled it, added 16, and obtained 50. Of what number did she think?

3. The continued height of a tower and flagstaff is 120 ft.; the height of the tower is 5 times that of the flagstaff. Find the height of each.

4. $\frac{2}{3}$ of the total height of a bridge pier is out of the water, and 10 ft. of the height is under water. What is the height of the pier?

5. When goods are sold at a gain of $\frac{1}{5}$ of their cost, what is the cost of goods which sell for $12?

6. A man's salary was increased by $\frac{1}{3}$ of itself; he then received $1600. What was his salary before the increase?

7. A merchant gained in one year an amount equal to $\frac{1}{4}$ of his capital. He then had $6250. How many dollars had he at the beginning of the year?

8. The area of Kansas is twice that of Ohio. The sum of their areas is 123,000 sq. mi. Find the area of each.

9. A freight train consisted of 48 cars. The number of closed cars was 6 more than twice the number of open cars. Find how many there were of each.

REVIEW

WRITTEN EXERCISES

1. $\frac{7}{15}$ of the distance from Boston to Cincinnati is 441 mi. Find the distance between these two cities.

2. The product of a certain number and 13 is 221. Write an equation expressing this fact, and find the number.

3. 3 times a certain number increased by 5 equals twice the number increased by 17. Find the number.

4. 3 times a number plus 3 equals $\frac{1}{3}$ of the number plus 27. Find the number.

5. In a recent year France mined twice as much coal as Russia, and together they produced 48 million tons. How many tons did each produce?

6. In a recent year the United States mined 12 times as much coal as Belgium, and together they produced 286 million tons. How many tons did each produce?

7. The length of a garden was 3 times its width, and the distance around it was 72 yd. Find its length and width.

8. If a tennis ball rebounds $\frac{2}{3}$ of the height from which it was dropped, from what height must it be dropped to rebound $3\frac{1}{2}$ ft.?

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define an *equation.* Sec. 36.
2. What are the *members* of an equation? Sec. 37.
3. What is an *identical equation?* Sec. 38.
4. What is a *conditional equation?* Sec. 39.
5. What is meant by *substitution?* Sec. 42.
6. What is an *unknown?* Sec. 43.
7. Explain the meaning of *satisfy an equation.* Sec. 44.
8. Define a *root* of an equation. Sec. 45.
9. Define *to solve* an equation. Sec. 46.
10. What are the *properties* used in solving equations? Sec. 48.
11. How is the correctness of the work in solving equations *tested?* Sec. 50.
12. How is the *solution* of a problem tested? Sec. 51.
13. How is a problem *stated* in the form of an equation? Sec. 52.
14. What is the sign of deduction? Sec. 53.

CHAPTER IV

RELATIVE NUMBERS

54. Numbers used in Algebra. The preceding work is much like that of arithmetic; in fact, it might be called *literal arithmetic.* We take up now a class of numbers belonging to algebra proper, and the following examples will illustrate them:

I. *Distances measured in opposite directions.*

1. 40 ft. up an elevator shaft + 20 ft. down the shaft is the same as how many feet up the shaft?

2. 20 ft. up + 40 ft. down is the same as how many feet down?

Similarly, to what is each of the following equivalent?

3. 40 rd. traveled to the right + 20 rd. traveled to the left.

4. 20 rd. traveled to the right + 20 rd. traveled to the left.

5. 20 rd. traveled to the right + 50 rd. traveled to the left.

II. *Rise and fall of temperature.*

1. Calling rise of temperature R and fall of temperature F $15° R + 10° F$ is the same as $(?)° R$.

Similarly:

2. $10° R + 15° F = ?$

3. $35° F + 45° R = ?$

4. $30° F + 15° R = ?$

5. $40° R + 40° F = ?$

III. *Amounts gained and lost.*

1. \$22 gain + \$26 loss = ? loss.

Similarly, using G for gain and L for loss:

2. $\$40\,G + \$30\,L = ?$

3. $\$17\,G + \$17\,L = ?$

4. $\$35\,L + \$20\,G = ?$

5. $\$45\,G + \$15\,L = ?$

55. Relative Numbers. In each of the preceding illustrations we have considered quantities which had two opposite directions, or **senses.** Numbers which measure quantities having opposite senses are called **relative numbers.**

56. In algebra we distinguish two opposite senses by calling one **positive** and the other **negative**. Either may be called positive, but the opposite to the positive is always called negative.

For example:

If distance *upward* is called *positive*, distance *downward* is called *negative*. If a *rise* of temperature is called *positive*, a *fall* is called *negative*.

57. Positive and Negative Number. A number that measures a quantity taken in the positive sense is called a **positive number**; one that measures a quantity taken in the negative sense is called a **negative number**.

ORAL EXERCISES

What must be taken as negative when each of the following is taken as positive?

Number of:

1. Feet to the right.
2. Miles southward.
3. Dollars gained.
4. Degrees upward.
5. Points won.
6. Pounds lifted.

58. Notation for Positive and Negative Numbers. Since a number added is offset by the same number subtracted, and relative numbers similarly offset each other, the signs + and − are used to designate positive and negative numbers respectively.

Thus:

$+3$ means 3 positive units, and denotes 3 units to be added.
-3 means 3 negative units, and denotes 3 units to be subtracted.

59. Signs of Character and Signs of Operation. Thus, in algebra the signs +, −, are used to indicate the *operations* of adding or subtracting numbers, and also to indicate the positive or negative *character* of numbers.

If it is necessary to distinguish a sign of character from a sign of operation, the former is put into a parenthesis with the number it affects.

Thus, $+8-(-3)$ means: positive 8 minus negative 3.

When no sign of character is expressed, the sign plus is understood.

Thus, $5-3$ means: positive 5 minus positive 3.
Similarly, $8a+9a$ means: positive $8a$ plus positive $9a$.

60. Signed Numbers. In algebra every number is understood to have either the sign + or the sign −. Consequently the numbers of algebra are often called **signed numbers.**

61. Absolute Value. The value of a signed number apart from its sign is called its **absolute,** or **numerical, value.**

Thus, 6° is the absolute value of 6° above zero or 6° below zero.
And 6 is the absolute value of either $+6$ or -6.

ORAL EXERCISES

Read the following in full, in accordance with Sec. 59:

1. $7-4$.
2. $-6-8$.
3. $-8+25$.
4. $-2+7$.
5. $9-(-3)$.
6. $14-(-6)$.
7. $-14-(+6)$.
8. $-6-(-8)$.
9. $7-15$.
10. $-7-(+2)$.
11. $6-8$.
12. $6+8$.
13. $-6+(-8)$.
14. $-4+(-9)$.
15. $15-(-9)$.

WRITTEN EXERCISES

Indicate, using the signs +, − :

1. The sum of positive 6 and positive 4.
2. The sum of positive a and negative b.
3. The difference of positive x and positive y.
4. The difference of negative 6 and positive 3.
5. The difference of negative a and positive b.
6. The sum of negative c and negative d.

62. Preparatory.

1. 4 points won + 3 points won = —— points won.
2. 4 points lost + 5 points lost = —— points lost.
3. 7° above zero + 9° above zero = —— degrees above zero.

63. Addition of Numbers having Like Signs. When the numbers added are positive, the sum is positive; and when they are negative, the sum is negative.

Therefore, to add either positive numbers or negative numbers, find the sum of their absolute values and prefix the corresponding sign.

ORAL EXERCISES

Add:

1. $\begin{array}{r}+2\\ \underline{+3}\end{array}$

2. $\begin{array}{r}-2\\ \underline{-3}\end{array}$

3. $\begin{array}{r}+4\\ \underline{+3}\end{array}$

4. $\begin{array}{r}-4\\ \underline{-3}\end{array}$

5. $\begin{array}{r}+6\\ \underline{+5}\end{array}$

6. $\begin{array}{r}-6\\ \underline{-5}\end{array}$

7. $\begin{array}{r}-4\\ \underline{-7}\end{array}$

8. $\begin{array}{r}-5\\ \underline{-8}\end{array}$

9. $\begin{array}{r}+2a\\ \underline{+3a}\\ (\quad)\,a\end{array}$

10. $\begin{array}{r}-2a\\ \underline{-3a}\\ (\quad)\,a\end{array}$

11. $\begin{array}{r}+4b\\ \underline{+3b}\\ (\quad)\,b\end{array}$

12. $\begin{array}{r}-4b\\ \underline{-3b}\end{array}$

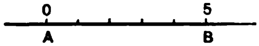
13. $\begin{array}{r}-6x\\ \underline{-5x}\end{array}$

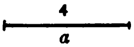
14. $\begin{array}{r}-5y\\ \underline{-8y}\end{array}$

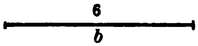
15. $\begin{array}{r}-10x\\ \underline{-\ \ 5x}\end{array}$

16. $\begin{array}{r}-12nx\\ \underline{-\ \ 8nx}\end{array}$

64. Number Pictures or Graphs. Numbers are often represented by lines. Such representations are called **graphs**.

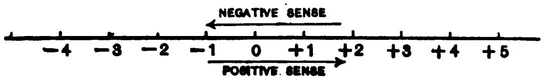






Thus, the line *AB* represents 5, the line *a* represents 4, and the line *b* represents 6.

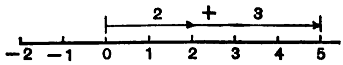
65. Graphical Addition. Positive and negative numbers may be arranged on a straight line as follows:



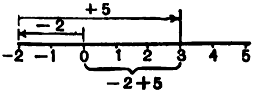
This arrangement is called the **number scale**, and it may be used to perform additions graphically.

EXAMPLES

1. To add + 2 and + 3: Let a moving point start at 0 and proceed 2 units in the positive direction (to the right), and from the place where it then is, proceed 3 units farther in the positive direction. The final position of the moving point will be distant 2 + 3 units from the starting point. That is, 2 + 3 = 5.

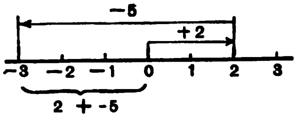


2. To add -2 and $+5$: Let the point proceed 2 units in the negative sense (toward the left), and from there 5 units in the positive sense. The final position is three positive units from the starting point. That is, $-2+5=3$.



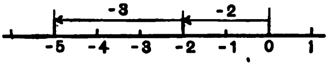
3. To add $+2$ and -5: Proceed 2 units to the right, and from there 5 units to the left. The final position of the moving point is three units to the left of the starting point. That is,

$$2+(-5)=-3.$$



4. Similarly, to add (-2) and (-3) proceed two units to the left, and from there three units to the left. The final position is 5 units to the left.

That is, (-2) and $(-3)=-5$.



WRITTEN EXERCISES

Add by means of the number scale as above:

1. $3+4$.
2. $8+3$.
3. $-2+5$.
4. $-2+8$.
5. $-2+9$.
6. $-9+5$.
7. $4+(-3)$.
8. $2+(-6)$.
9. $7+(-3)$.
10. $-2+(-5)$.
11. $-3+(-3)$.
12. $-4+(-7)$.

66. PREPARATORY.

1. Add 5 and -3.

Regard 5 as made up of $+3$ and $+2$; then

$$\left.\begin{array}{r} +3+2 \\ \underline{-3\qquad} \\ 0+2 \end{array}\right\} \text{ or } \left\{\begin{array}{r} 5 \\ \underline{-3} \\ 2 \end{array}\right.$$

That is, the -3 offsets $+3$ of the $+5$, and the sum is $+2$.

2. Add -8 and 6.

Regard -8 as made up of -6 and -2; then

$$\left.\begin{array}{r} -6-2 \\ \underline{+6\qquad} \\ 0-2 \end{array}\right\} \text{ or } \left\{\begin{array}{r} -8 \\ \underline{6} \\ -2 \end{array}\right.$$

That is, the $+6$ offsets -6 of the -8, and the sum is -2.

Zero, as in arithmetic, means the difference between two equal numbers like $3-3$; hence, in accordance with Sec. 58, $3+(-3)=0$.

Thus the addition of zero has no effect; $6+0=6$, or $-6+0=-6$.

67. Addition of Numbers with Unlike Signs. In adding a positive and a negative number, a positive unit and a negative unit offset each other.

Therefore, *to add a positive and a negative number find the difference of their absolute values and prefix to it the sign of the number having the greater absolute value.*

ORAL EXERCISES

State the sums:

1. 7 negative units + 4 positive units.
2. 7 negative units + 12 positive units.
3. 8 negative units + 7 positive units.
4. 9 positive *a*'s + 9 negative *a*'s.
5. 10 positive *x*'s + 15 negative *x*'s.

WRITTEN EXERCISES

Add, separating the numbers as in Sec. 66:

1. $\begin{array}{r} 5 \\ -2 \\ \hline \end{array}$
2. $\begin{array}{r} 6 \\ -4 \\ \hline \end{array}$
3. $\begin{array}{r} 8 \\ -5 \\ \hline \end{array}$
4. $\begin{array}{r} 10 \\ -7 \\ \hline \end{array}$
5. $\begin{array}{r} -3 \\ 7 \\ \hline \end{array}$
6. $\begin{array}{r} -5 \\ 9 \\ \hline \end{array}$
7. $\begin{array}{r} -6 \\ 12 \\ \hline \end{array}$
8. $\begin{array}{r} -11 \\ 20 \\ \hline \end{array}$
9. $\begin{array}{r} -9 \\ 3 \\ \hline \end{array}$
10. $\begin{array}{r} -6 \\ 2 \\ \hline \end{array}$
11. $\begin{array}{r} -12 \\ 8 \\ \hline \end{array}$
12. $\begin{array}{r} -17 \\ 9 \\ \hline \end{array}$
13. $\begin{array}{r} -40 \\ 30 \\ \hline \end{array}$
14. $\begin{array}{r} -25 \\ 17 \\ \hline \end{array}$
15. $\begin{array}{r} 43 \\ -55 \\ \hline \end{array}$
16. $\begin{array}{r} 39 \\ -44 \\ \hline \end{array}$
17. $\begin{array}{r} -9a \\ +5a \\ \hline \end{array}$
18. $\begin{array}{r} +9b \\ -5b \\ \hline \end{array}$
19. $\begin{array}{r} +9b \\ -9b \\ \hline \end{array}$
20. $\begin{array}{r} -15p \\ +10p \\ \hline \end{array}$
21. $\begin{array}{r} -3x \\ +9x \\ \hline \end{array}$
22. $\begin{array}{r} +\ 8x \\ -13x \\ \hline \end{array}$
23. $\begin{array}{r} -12ab \\ +10ab \\ \hline \end{array}$
24. $\begin{array}{r} -8mn \\ +9mn \\ \hline \end{array}$
25. $\begin{array}{r} +4xy \\ -8xy \\ \hline \end{array}$

68. Preparatory.

1. $5 + ? = 8.$

2. $5 + ? = 3.$

3. $-5 + ? = -7.$

4. $-5 + ? = 3.$

5. Question 1 may be read "8 less 5 are how many?" Read the questions in 2, 3, and 4, and state the difference in each case.

69. Subtraction of Positive and Negative Numbers. The difference is the number which added to the subtrahend produces the minuend.

Thus: 6 less $-4 = 10$ because $-4 + 10 = 6.$
-15 less $-7 = -8$ because -7 plus $-8 = -15.$
$4a$ less $-2a = 6a$ because $-2a + 6a = 4a.$

The terms **subtrahend** and **minuend** are used as in arithmetic, the former to mean the number taken away and the latter the number from which the subtrahend is taken.

ORAL EXERCISES

State the numbers to fill the blanks:

1. $\begin{cases} 8 + (\ ) = 15, \\ 15 - 8 = (\ ). \end{cases}$

2. $\begin{cases} 8 - (\ ) = 2, \\ 2 - 8 = (\ ). \end{cases}$

3. $\begin{cases} 12 - (\ ) = -1, \\ -1 - 12 = (\ ). \end{cases}$

4. $\begin{cases} -7 + (\ ) = 3, \\ 3 - (-7) = (\ ). \end{cases}$

5. $\begin{cases} -6 + (\ ) = 8, \\ 8 - 14 = (\ ). \end{cases}$

6. $\begin{cases} -5 + (\ ) = 20, \\ 20 - 25 = (\ ). \end{cases}$

7. $\begin{cases} -6 + (\ ) = -9, \\ -9 - (-3) = (\ ). \end{cases}$

8. $\begin{cases} -11 + (\ ) = -15, \\ -15 - (-4) = (\ ). \end{cases}$

9. $\begin{cases} 14 + (\ ) = -7, \\ -7 - (-21) = (\ ). \end{cases}$

10. $\begin{cases} 15 + (\ ) = -5, \\ -5 - (-20) = (\ ). \end{cases}$

11. $20 - 8 = (\ ).$

12. $8 - 20 = (\ ).$

13. $3 - (-9) = (\ ).$

14. $16 - (-5) = (\ ).$

15. $-17 - 13 = (\ ).$

16. $-15 - (-5) = (\ ).$

70. We have shown in Sec. 58 that a negative and a positive unit offset each other. Hence, to subtract 1 is the same as to add -1, and *vice versa;* and to subtract any number a is to add $-a$, and *vice versa.*

Thus: 7 less $5 = 2$ is the same as 7 plus $-5 = 2$,
or, 7 less $-5 = 12$ is the same as 7 plus $5 = 12$.

By its meaning the subtraction of zero has no effect. (Sec. 66.)

71. *To subtract a number, change its sign and add it.*

ORAL EXERCISES

Find these differences by adding:

1. $\begin{array}{r} 6 \\ -3 \\ \hline \end{array}$

2. $\begin{array}{r} 3 \\ -8 \\ \hline \end{array}$

3. $\begin{array}{r} 5 \\ -3 \\ \hline \end{array}$

4. $\begin{array}{r} 10 \\ -5 \\ \hline \end{array}$

5. $\begin{array}{r} 12 \\ -9 \\ \hline \end{array}$

6. $\begin{array}{r} 15 \\ -15 \\ \hline \end{array}$

7. $\begin{array}{r} 7 \\ -15 \\ \hline \end{array}$

8. $\begin{array}{r} 8 \\ -17 \\ \hline \end{array}$

9. $\begin{array}{r} 7a \\ 4a \\ \hline (\quad)a \end{array}$

10. $\begin{array}{r} 4a \\ 7a \\ \hline (\quad)a \end{array}$

11. $\begin{array}{r} 7a \\ -4a \\ \hline \end{array}$

12. $\begin{array}{r} -4a \\ 7a \\ \hline \end{array}$

13. $\begin{array}{r} -4a \\ -7a \\ \hline (\quad)a \end{array}$

14. $\begin{array}{r} 8x \\ 7x \\ \hline (\quad)x \end{array}$

15. $\begin{array}{r} 7x \\ 8x \\ \hline \end{array}$

16. $\begin{array}{r} -7x \\ -8x \\ \hline \end{array}$

Perform the indicated operations:

17. $13-(-5)+8$.

18. $13-(-6)+(-4)$.

19. $-15+(-8)-(+22)$.

20. $-17-9-(-20)$.

72. Preparatory.

1. A man earned \$3 on Monday and \$3 on Tuesday. How many dollars did he earn in the two days?

2. To multiply 3 by 2 is to take 3 how many times as the addend?

3. A man lost \$3 on Monday and \$3 on Tuesday. How many dollars did he lose in the two days?

4. To multiply -3 by 2 is to take -3 how many times as an addend?

73. Multiplication of Positive and Negative Numbers. *Multiplication by a positive integer means taking the multiplicand as an addend as many times as there are units in the multiplier.*

Correspondingly, *multiplication by a negative integer means taking the multiplicand as a subtrahend as many times as there are units in the multiplier.*

For example:

4 multiplied by $3 = 4 + 4 + 4 = 12$.
-4 multiplied by $3 = -4 + (-4) + (-4) = -12$.
4 multiplied by $-3 = -4 - 4 - 4 = -12$.
-4 multiplied by $-3 = -(-4) - (-4) - (-4) = +4 + 4 + 4 = 12$.

The numerical value of the product is the product of the numerical values of the factors.

74. The law of signs in multiplication, which applies to integral and fractional numbers alike, may be stated thus: If both factors are positive or if both are negative, their product is *positive*. If one is positive and the other negative, their product is *negative*.

In symbols: + times + = + — times + = —.
— times — = + + times — = —.

75. This law is easily remembered in the form:

The product of two factors of like signs is positive, and of two factors of unlike signs is negative.

ORAL EXERCISES

State the product in each of the following:

1. $+5 \cdot +3$.
2. $-5 \cdot +3$.
3. $+5 \cdot -3$.
4. $-5 \cdot -3$.
5. $-7 \cdot +7$.
6. $+8 \cdot -9$.
7. $-7 \cdot -6$.
8. $-8 \cdot -8$.
9. $-7 \cdot -11$.
10. $-12 \cdot +6$.
11. $+5 \cdot -11$.
12. $+6 \cdot -\frac{2}{3}$.
13. $-8 \cdot +\frac{1}{4}$.
14. $+3 \cdot -4 \cdot +5$.
15. $-5 \cdot +2 \cdot -3$.
16. $+14 \cdot +\frac{1}{2}$.
17. $+2 \cdot -3 \cdot -4$.
18. $-4 \cdot -5 \cdot -3$.

76. Division of Positive and Negative Numbers. We have seen in arithmetic that

$$\text{Quotient} \times \text{Divisor} = \text{Dividend}.$$

Thus, we know that $24 \div 8$ is 3, because $3 \times 8 = 24$.

Applying this relation to division with positive and negative numbers, we can determine the sign of the quotient. There are four cases, as follows:

$+24 \div +8 = (\ )$; $-24 \div -8 = (\ )$; $+24 \div -8 = (\ )$; $-24 \div +8 = (\ )$.

According to Sec. 74:

$+24 \div +8 = +3$ **because** $+3 \times +8 = +24$.
$-24 \div -8 = +3$ **because** $+3 \times -8 = -24$.
$+24 \div -8 = -3$ **because** $-3 \times -8 = +24$.
$-24 \div +8 = -3$ **because** $-3 \times +8 = -24$.

In symbols:

$$\frac{+}{+} = +;\quad \frac{-}{-} = +;\quad \frac{+}{-} = -;\quad \frac{-}{+} = -.$$

77. This law is easily remembered in the form:

The quotient of two numbers of like signs is positive, and the quotient of two numbers of unlike signs is negative.

ORAL EXERCISES

State the quotient:

1. $-6 \div +3$.
2. $-12 \div +4$.
3. $+16 \div -4$.
4. $+20 \div -5$.
5. $+15 \div -3$.
6. $-8 \div -8$.
7. $-8 \div +8$.
8. $-15 \div -5$.
9. $-36 \div -6$.
10. $-35 \div +7$.
11. $+10 \div -4$.
12. $+18 \div -12$.
13. $20 \div -4$.
14. $30 \div -10$.
15. $-42 \div +7$.

Perform the indicated operations:

16. $6(-3) \div 9$.
17. $4(-6) \div -12$.
18. $18 \div (-3) \div -2$.
19. $4(-9) - 3 \times 3 - (18 \div -2)$.
20. $6 \times 9 - 4 \times 9 - (-24 \div -4)$.

78. The Greater of Two Numbers. Of two given numbers that one is the greater which can be produced by adding a positive number to the other. The other number is the less.

For example:

11 is greater than 8 because it is necessary to add $+3$ to 8 to make 11.

7 is greater than -2 because it is necessary to add $+9$ to -2 to make 7.

-4 is greater than -9 because it is necessary to add $+5$ to -9 to make -4.

79. The symbol $>$ is read "is greater than," and $<$ is read "is less than."

For example:

$8 > 2$ is read "8 is greater than 2."
$-1 > -5$ is read "-1 is greater than -5."
$-5 < -1$ is read "-5 is less than -1."

The terms "algebraically greater" and "algebraically less" are used when positive and negative numbers are compared. The terms "numerically greater" and "numerically less" apply to absolute values.

For example:

4 is greater than -9, but 4 is numerically less than -9.

-6 is greater than -15, but -15 is numerically greater than -6.

-2 is algebraically greater than -12, but numerically less than -12.

ORAL EXERCISES

Read the following and state why each is correct:

1. $7 > 5$.
2. $4 > -8$.
3. $-2 > -5$.
4. $0 > -7$.
5. $3 < 5$.
6. $-4 < 2$.
7. $-1 < 0$.
8. $-8 < -6$.

From the following list select the numbers that are:

a. Greater than 6.
b. Less than -5.
c. Greater than -4.
d. Numerically greater than -4.
e. Numerically greater than 6.
f. Numerically less than -5.

9. 7.
10. -10.
11. -8.
12. -18.
13. -1.
14. 0.
15. -3.
16. 2.
17. -4.
18. 5.
19. -6.
20. $-\frac{1}{2}$.

25. State the absolute value of each of the numbers in Exercises 9–20.

WRITTEN EXERCISES

Determine which is the greater in each of the following pairs of numbers, and write the relation by use of the sign $>$:

1. $8, 6$.
2. $3, 4$.
3. $-5, 6$.
4. $-6, 5$.
5. $-6, -5$.
6. $6, -9$.
7. $0, 10$.
8. $0, -10$.
9. $-4, -2$.

80. Processes with inequalities.

1. Addition:

(1)
$$\begin{array}{rl} & 4>3 \\ & 5>2 \\ \hline \text{Adding,} & 4+5>3+2 \end{array}$$

(2)
$$\begin{array}{rl} & 3<4 \\ & -5<-2 \\ \hline \text{Adding,} & 3-5<4-2 \end{array}$$

That is, if two inequalities of the same kind are added the result will be an inequality of the same kind, but the sum of two inequalities of different kinds may result in an equality or in an inequality of either kind. Similarly, the subtraction of inequalities is uncertain.

2. Multiplication:

(1) $3<5$ then $2\cdot 3<2\cdot 5$

(2) $-2>-5$ then $3(-2)>3(-5)$

If the members of an inequality be multiplied by any number not zero or negative, the result will be an inequality of the same kind. If the multiplier is negative, the result will be an inequality of the opposite kind.

ORAL EXERCISES

Add:

1. $$\begin{array}{r} 3>2 \\ \underline{4>3} \end{array}$$

2. $$\begin{array}{r} 3>2 \\ \underline{-4>-5} \end{array}$$

3. $$\begin{array}{r} -3<-2 \\ \underline{-7<-5} \end{array}$$

4. Multiply: $5>2$ by 3; also by -2; also by -1.

5. Multiply: $a<b$ by 3; also by -2; also by -1.

REVIEW

ORAL EXERCISES

1. The temperature was $-8°$ at 6 o'clock and $+5°$ at 9 o'clock. How many degrees did it rise in this interval?

2. A ship sailed on a meridian from Lat. $+12°$ to Lat. $-2°$: Through how many degrees did it sail?

Read in full:

3. $11 + 18$. 6. $3 + (-2)$. 9. $xy - (-xy)$.
4. $14 - 9$. 7. $p + (-q)$. 10. $ab - (-ab)$.
5. $-2 - (+3)$. 8. $-3a - (+2b)$. 11. $mn + (-2m)$.

Add:

12. $\begin{array}{r} -6 \\ \underline{-9} \end{array}$ 14. $\begin{array}{r} -16 \\ \underline{7} \end{array}$ 16. $\begin{array}{r} -\ 8 \\ \underline{26} \end{array}$ 18. $\begin{array}{r} -18 \\ \underline{36} \end{array}$

13. $\begin{array}{r} 15 \\ \underline{-9} \end{array}$ 15. $\begin{array}{r} 23 \\ \underline{-9} \end{array}$ 17. $\begin{array}{r} 40 \\ \underline{-15} \end{array}$ 19. $\begin{array}{r} 33 \\ \underline{-13} \end{array}$

Subtract:

20. $\begin{array}{r} 9 \\ \underline{-5} \end{array}$ 22. $\begin{array}{r} -15 \\ \underline{-\ 6} \end{array}$ 24. $\begin{array}{r} -14 \\ \underline{8} \end{array}$ 26. $\begin{array}{r} 8 \\ \underline{-16} \end{array}$

21. $\begin{array}{r} -8 \\ \underline{4} \end{array}$ 23. $\begin{array}{r} -32 \\ \underline{-\ 3} \end{array}$ 25. $\begin{array}{r} -\ 3 \\ \underline{15} \end{array}$ 27. $\begin{array}{r} -14 \\ \underline{-14} \end{array}$

Multiply:

28. $\begin{array}{r} 6 \\ \underline{-5} \end{array}$ 30. $\begin{array}{r} -4 \\ \underline{-8} \end{array}$ 32. $\begin{array}{r} -6 \\ \underline{10} \end{array}$ 34. $\begin{array}{r} -12 \\ \underline{3} \end{array}$

29. $\begin{array}{r} -8 \\ \underline{3} \end{array}$ 31. $\begin{array}{r} -15 \\ \underline{3} \end{array}$ 33. $\begin{array}{r} -11 \\ \underline{-\ 3} \end{array}$ 35. $\begin{array}{r} -10 \\ \underline{0} \end{array}$

Divide:

36. -15 by 3. 38. 18 by -6.
37. -20 by -4. 39. 40 by -10.

WRITTEN EXERCISES

1. Write the sum of positive a and negative b.

Indicate, by using the signs $+$, $-$:

2. \$15 lost plus \$10 gained.

3. The sum of positive x and negative y.

4. The sum of negative m and positive n.

5. The difference of positive m and negative n.

6. The difference of positive x and negative y.

Perform the indicated operations:

7. $-6+5-4(-3)$.

8. $9-3(-3)+6\cdot 0$.

9. $15\div -5+(3\cdot -5)$.

10. $16\div -2-(4\cdot -3)$.

11. $12\cdot -3+24-(6\cdot -3)$.

12. $17(2-8)-(70\div -7)$.

13. $(-5\cdot 15-3)\div(15-4\cdot 6)$.

14. $-6a-2(-4a)-(8a\div -2)$.

15. Multiply $-2x>-5x$ by -3.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What are *relative numbers?* *Negative numbers?* Sec. 56.

2. What signs are used to indicate the *positive and negative character* of numbers respectively? Sec. 59.

3. State the rules that determine when the signs $+$, $-$, indicate *signs of operation.* Sec. 59.

4. Why are the numbers of algebra called *signed* numbers? Sec. 60.

5. Define *absolute,* or *numerical* value. Sec. 61.

6. If the addends have *like* signs, what is the sign of their sum? Sec. 63.

7. In adding two numbers with *unlike* signs, how is the sign of the sum found? Sec. 67.

8. How are *algebraic* numbers subtracted? Secs. 70, 71.

9. State the *law of signs in multiplication.* Sec. 75.

10. State the *law of signs in division.* Sec. 77.

11. Define the *greater* of two numbers; the *less.* Sec. 78.

12. Define *numerically greater; algebraically greater.* Sec. 79.

13. What is the result of adding inequalities of the *same kind?* Of *different kind?* Sec. 80.

14. What is the result of multiplying both members of an inequality by a *positive* number? By a *negative* number?

HISTORICAL NOTE

For the first use of negative number we must turn to the Brahmin schools of India. The Hindoo priests were clever mathematicians, and tradition relates that the great reformer, Buddha, in his youth won the maiden he loved by solving a difficult set of problems. Hindoo scholars did much to develop algebra, and the writings of Bhaskara, who lived about 1150 A.D., is a summary of their work. The poetic tendency of the Hindoos affected all their thinking, and they expressed their mathematics in flowery language and in verse. Bhaskara called one part of his work *Lilavati* ("noble science"), and proposed many problems like the following: "The square root of half the number of bees in a swarm flew to a jasmine bush; $\frac{8}{9}$ of the whole swarm remained behind; one bee, allured by the sweet odor of a lotus flower, became entangled in it while his excited mate lingered about. Tell me the number of bees." (Answer, 72.)

The Hindoos were the first to explain positive and negative numbers by reference to debits and credits, and the modern interpretation of these numbers by opposite directions on a straight line was not unknown to them. They discovered that the solution of certain equations gave negative roots, but, strange as it may seem, they rejected them and negative numbers received no further acceptance until Descartes fully interpreted them (1637). They had so long remained a mystery that they were known as "absurd" or "fictitious" numbers. Even as late as 1545 Cardan called them *numeræ fictæ.* Hence, we may in a measure be forgiven for calling negative number artificial in comparison with positive number. It is unfortunate, however, that the negative number came down to us labeled "artificial" or "unreal" number, because it is just as real as the positive number. Both kinds have to exist, or neither could exist. Both positive and negative numbers are concrete. They are denominate, or named numbers, just as much as 3 feet, or 7 quarts. It is quite as natural to think three positive and three negative units as it is to think three dollars' gain and three dollars' loss. When the unit, *negative one,* has been defined to be an absolute one taken in the opposite sense to positive one, we can as easily count or reckon with negative numbers as with positive numbers.

CHAPTER V

ADDITION

ADDITION OF MONOMIALS

81. Algebraic Sum. The result of adding numbers some or all of which are negative is called their **algebraic sum.**

82. Like Terms. Terms or monomials that have the same literal factors, are called **like terms** or **like monomials.**

For example, the following are pairs of like terms:

ab and ab; $5a$ and $-3a$; $4ab$ and $\frac{7}{8}ab$; $2a^2b$ and $\frac{1}{3}a^2b$.

Like terms in algebra correspond to numbers in arithmetic having a common factor.

For example, 5 × *7* and 3 × *7*; or 7 × *15* and 6 × *15*.

ORAL EXERCISES

From the following list select terms like the first; like the second; like the third:

1. ab^3x.
2. cy^3.
3. ab^2.
4. mp.
5. $2b^2x^2$.
6. amp.
7. $-\frac{2}{3}ab^2$.
8. $5ab^3x$.
9. $-4cy^3$.
10. $-6\frac{1}{5}ab^2$.
11. ax.
12. $24cy^2$.

83. Terms are *alike in any letter* if they contain the same power of that letter.

Thus, $2ax^2$ and bx^2 are alike in x.

Which of the terms of Exercises 1–12 are alike in a? In b? In x?

84. *To add two like terms or monomials, prefix to their like factor the sum of their coefficients.*

In arithmetic we add 3×7 and 5×7 by adding 3 and 5 and writing 8×7.

Similarly, in $3\,ax + 5\,ax = 8\,ax$, we add 3 and 5 and write $8\,ax$.

In $-4\,ab^2 + 6\,ab^2$ the like factor is ab^2; hence, adding the coefficients -4 and 6 the whole sum is $+2\,ab^2$.

The addition of dissimilar terms can only be indicated.

ORAL EXERCISES

Add:

1. $-5x$ $-2x$	3. $8y$ $-5y$	5. $-15y^2$ $-10y^2$	7. $-6a^2b$ $-3a^2b$
2. $-6x$ $2x$	4. $6a^2b$ $2a^2b$	6. $-10xy$ $5xy$	8. $16mn$ $-9mn$

9. $-7\,ab + (-3\,ab) = (\quad)ab.$

10. $3\,abc + (-5\,abc) = (\quad)abc.$

11. $.7\,x + (-.3\,x) = (\quad)x.$

12. $8\,a^3 + (-5\,a^3) = (\quad)a^3.$

85. To add several like terms, *combine them in order, or add the positive and the negative terms separately, and then combine these two sums.*

$$\begin{array}{r} 3a \\ -5a \\ 12a \\ \underline{-6a} \\ 4a \end{array}$$

Thus, in the column on the left, adding upward, $-6\,a + 12\,a = +6\,a$, then $6\,a + (-5\,a) = a$, and finally $a + 3\,a = 4\,a$; or the sum of the positive terms is $15\,a$, the sum of the negative terms is $-11\,a$, and the sum of these two is $4\,a$.

WRITTEN EXERCISES

Add:

1. $7b$ $-3b$ $20b$	2. $12y$ $-9y$ $5y$	3. $4m^3$ $-m^3$ $25m^3$	4. $-2x$ $8x$ $-9x$	5. $8xy^2$ $10xy^2$ $-20xy^2$

6. $\begin{array}{r} 4\,s \\ -12\,s \\ 6\,s \\ \underline{-9\,s} \end{array}$ **7.** $\begin{array}{r} -x^2y \\ 9\,x^2y \\ -2\,x^2y \\ \underline{4\,x^2y} \end{array}$ **8.** $\begin{array}{r} 7\,b \\ 8\,b \\ -20\,b \\ \underline{3\,b} \end{array}$ **9.** $\begin{array}{r} t^2 \\ 2\,t^2 \\ -5\,t^2 \\ \underline{25\,t^2} \end{array}$ **10.** $\begin{array}{r} 20\,w \\ 30\,w \\ -19\,w \\ \underline{11\,w} \end{array}$

11. $3\,x - 8\,x + 15\,x = ?$ **12.** $6\,x + 8\,x - 3\,x - 4\,x = ?$

13. $ax^2 - \frac{1}{2}\,ax^2 + \frac{3}{2}\,ax^2 = (\ \ )\,ax^2.$

14. $\frac{1}{3}\,ab^2 + \frac{1}{6}\,ab^2 - \frac{4}{3}\,ab^2 + \frac{5}{6}\,ab^2 = (\ \ )\,ab^2.$

15. $-14\,x + 23\,x + 99\,x = ?$

16. $40\,a - 75\,a + 89\,a = ?$

17. $12\,ab - 18\,ab + 75\,ab = ?$

18. $13\,xy - 50\,xy + 113\,xy = ?$

19. $-x^2 + 4\,x^2 - 10\,x^2 = ?$

20. $-xy - 40\,xy + 9\,xy = ?$

21. $-mn^2 + 12\,mn^2 - 15\,mn^2 = ?$

22. $15\,xy^2 + 33\,xy^2 - 48\,xy^2 = ?$

23. $4\,z^2 - 52\,z^2 + 109\,z^2 - 12\,z^2 = ?$

24. $29\,x^2 - 43\,x^2 + 87\,x^2 = ?$

25. $-20\,pq - 100\,pq + 7\,pq = ?$

26. $-y^2 + 4\,y^2 - 117\,y^2 + 3\,y^2 = ?$

27. $-12\,pqr^2 - 17\,pqr^2 - 53\,pqr^2 = ?$

Add the terms in x and the other terms separately:

28. $17\,x - 5\,x + 3\,x + 2 - 11\,x - 4\,x + 10\,x - 27.$

29. $22\,x - 3\,x - 6 - 4\,x + 5\,x - x - 2\,x - 9.$

30. $2\,x + 3\,a + 15\,x - 12\,x + 6\,a - 24\,a.$

ADDITION OF POLYNOMIALS

86. The addition of polynomials is similar to that of denominate numbers.

For example:

	Denominate Numbers		Polynomials
Just as:	3 bu. 4 qt.	so:	$3\,b + 4\,q$
plus	5 bu. 3 qt.	plus	$5\,b + 3\,q$
equals	8 bu. 7 qt.	equals	$8\,b + 7\,q$

ORAL EXERCISES

1. Add: $\begin{array}{l} \text{4 mi. 3 rd. 2 ft.} \\ \underline{\text{6 mi. 7 rd. 8 ft.}} \end{array}$ also, $\begin{array}{r} 4m+3r+2f \\ \underline{6m+7r+8f} \end{array}$

State the numbers to fill the blanks in the following additions:

2.

$$\begin{array}{r} 2a+4b \\ \underline{3a+5b} \\ (\ )a+(\ )b \end{array}$$

3.

$$\begin{array}{r} 2a+\ \ b+\ \ c \\ \underline{5a+3b+2c} \\ (\ )a+(\ )b+(\ )c \end{array}$$

87. To Add Polynomials: *Arrange the like terms in columns and add as in the case of monomials, using the signs obtained as the signs of the result.*

For example:

NOT ARRANGED

$$\begin{array}{r} a+c+b \\ \underline{-3b+a+c} \end{array}$$

ARRANGED

$$\begin{array}{r} a+\ \ b+c \\ \underline{a-3b+c} \end{array}$$

Here the first column is $+a+a=+2a$; the second column is $+b-3b=-2b$; the third column is $+c+c=+2c$. The terms thus obtained with their signs, namely, $2a-2b+2c$, constitute the sum of the polynomials.

88. Commutative Law of Addition. *The sum of two or more terms of a polynomial is the same in whatever order the terms are taken.*

The rearrangement of the terms of polynomials before adding is an application of this law.

WRITTEN EXERCISES

Add:

1. $\begin{array}{r} 5a-3b \\ \underline{a-\ \ b} \end{array}$

2. $\begin{array}{r} 2a-6c \\ \underline{3c+2a} \end{array}$

3. $\begin{array}{r} \frac{1}{2}x-\frac{1}{3}y \\ \underline{y+\frac{2}{3}x} \end{array}$

4. $\begin{array}{r} 3m-1.1n \\ \underline{6m-\ \ .9n} \end{array}$

5. $\begin{array}{r} \frac{2}{3}p+1\frac{4}{9}q \\ \underline{\frac{1}{3}q+\ \frac{5}{9}p} \end{array}$

6. $\begin{array}{r} -7x+\ \ 4y \\ \underline{9x-10y} \end{array}$

7. $$\begin{array}{r} x^2+2y^2-\ \ z^2 \\ z^2-3y^2+2x^2 \\ \hline \end{array}$$

8. $$\begin{array}{r} at^2-6at+\ \ c \\ 2at-8at^2+2c \\ \hline \end{array}$$

9. $$\begin{array}{r} m^2+\ \ m+1 \\ m-2m^2-8 \\ \hline \end{array}$$

10. $$\begin{array}{r} p^2+\ \ p+8 \\ p^2+9p+6 \\ \hline \end{array}$$

11. $$\begin{array}{r} x^3+6x^2-\ \ 9x \\ 2x^3+8x^2-10x \\ \hline \end{array}$$

12. $$\begin{array}{r} 17p^2-\ \ pq+q^2 \\ -6p^2+2pq-q^2 \\ \hline \end{array}$$

13. $$\begin{array}{r} 2a+7b+11c \\ 2b+\ \ 9c \\ \hline \end{array}$$

14. $$\begin{array}{r} 12x+\ \ 8y+17z \\ 9x+12z+13y \\ \hline \end{array}$$

15. $$\begin{array}{r} \frac{1}{8}p+\ \ \frac{2}{3}q+\frac{3}{7}r \\ 7\frac{1}{8}p+9\frac{1}{3}q+\ \ r \\ \hline \end{array}$$

16. $$\begin{array}{r} \frac{1}{5}r-\ \ \frac{3}{4}s+4t \\ \frac{4}{5}r+1\frac{1}{2}s-\frac{1}{2}t \\ \hline \end{array}$$

89. PREPARATORY.

Find the value of each expression when each letter = 1:

1. $a+2b$.
2. $a+b+c$.
3. $c+4d-2a$.
4. $2a-2b$.
5. $b+2c-a$.
6. $2a+3b+3c$.
7. $3a-c$.
8. $a+d+3c$.
9. $3b+c+a$.

90. Test of Addition. To test the work of addition, substitute unity for the letters. The value of the sum must equal the sum of the values of the expressions.

In practice the work and test are written as follows:

$$\begin{array}{rcr} \text{SOLUTION} & \quad & \text{TEST} \\ 2a-5b & & -3 \\ 4a+4b & & +8 \\ \hline 6a-\ \ b & & +5 \end{array}$$

The use of unity tests the coefficients including their signs, but does not check mistakes made in writing the literal parts. Such errors, however, rarely occur, and are easily discovered by inspection. Numbers other than unity are apt to make the work of checking too complicated.

WRITTEN EXERCISES

Add and test:

1. $$\begin{array}{r} a+\ \ 3b \\ 11a+10b \\ \hline \end{array}$$

2. $$\begin{array}{r} 6c+\ \ d \\ 3c+2d+e \\ \hline \end{array}$$

3. $$\begin{array}{r} 12t-\ \ 6t^2 \\ 8t+12t^2 \\ \hline \end{array}$$

4. $$\begin{array}{rrr} 4x & & +z \\ 2z & +y & +x \\ \hline \end{array}$$

5. $$\begin{array}{rr} 40m & +n \\ 5m & -39n \\ \hline \end{array}$$

6. $$\begin{array}{rr} 17a & +4b \\ 3a & -16b \\ \hline \end{array}$$

7. $$\begin{array}{rr} 6a & +9b \\ 4a & -b \\ \hline \end{array}$$

8. $$\begin{array}{rr} 12a & +5b \\ 6a & -3b \\ \hline \end{array}$$

9. $$\begin{array}{rr} -15x & +12 \\ -8x & -3 \\ \hline \end{array}$$

10. $$\begin{array}{rr} 5x^2 & +3x \\ -3x & +7x^2 \\ \hline \end{array}$$

11. $$\begin{array}{rr} 4xy & -2z^2 \\ 5xy & +10z^2 \\ \hline \end{array}$$

12. $$\begin{array}{rr} 40 & -\frac{1}{2}gt^2 \\ 50 & +\frac{3}{2}gt^2 \\ \hline \end{array}$$

13. $$\begin{array}{rrr} 45m & -2n & +q \\ 5m & +3n & +q \\ \hline \end{array}$$

14. $$\begin{array}{rrr} 3a^2 & -5a & +1 \\ 4a & +8a^2 & -3 \\ \hline \end{array}$$

15. $$\begin{array}{rrr} 5x & +y & -z \\ 3x & -7y & +8z \\ \hline \end{array}$$

16. $$\begin{array}{rrr} \frac{1}{2}x & +\frac{1}{3}y & +.9z \\ \frac{1}{2}x & +1\frac{2}{3}y & +1.1z \\ \hline \end{array}$$

17. $$\begin{array}{rrr} 1.1a & -8.9b & +c \\ 3.9a & +.5b & -5c \\ \hline \end{array}$$

18. $$\begin{array}{rrr} x^2 & +xy & +3y^2 \\ 6x^2 & +10xy & +5y^2 \\ \hline \end{array}$$

19. $$\begin{array}{rrrr} x^2 & +x^2y & +8y^2 & \\ & 4x^2y & +y^2 & +z^2 \\ \hline \end{array}$$

20. $$\begin{array}{rr} 5a & -7b \\ 3a & +10b \\ -6a & +18b \\ -7a & +12b \\ \hline \end{array}$$

21. $$\begin{array}{rr} x^2 & -5x \\ -4x^2 & +3x \\ -12x^2 & +x \\ 15x^2 & -12x \\ -10x^2 & +16x \\ \hline \end{array}$$

22. $$\begin{array}{rrr} a & +2b & +3c^2 \\ 2a & & +66c^2 \\ & 9b & +c^2 \\ a & +b & +c^2 \\ 5a & -b & \\ \hline \end{array}$$

23. $$\begin{array}{rrrr} & p & +3q & \\ m & +3p & +q & +r \\ 5m & +2p & & +6r \\ & & 10q & +5r \\ \hline \end{array}$$

24. $$\begin{array}{rrr} a^2 & +4 & -a \\ 5 & -3a^2 & +2a \\ 6a^2 & +3a & -5 \\ 4a & -7a^2 & -2 \\ 8a & -12 & -15a^2 \\ \hline \end{array}$$

25. $$\begin{array}{rrr} 4g & +3v & -7x \\ 5x & +2y & -4v \\ 2y & -8g & -7v \\ 13v & -11x & +2g \\ 12x & -15g & +4y \\ \hline \end{array}$$

26. A dealer bought at one time 3 kinds of coal, 50 a tons of the first kind, 10 b tons of the second kind, and 12 c tons of the third; at another time he bought 75 a tons, 15 b tons, 10 c tons respectively of the same kinds. How many tons did he buy in all?

REVIEW

ORAL EXERCISES

Add:

1. $$\begin{array}{r} -6ab \\ -8ab \\ \hline \end{array}$$

2. $$\begin{array}{r} 12ac^2 \\ -\ \ 8ac^2 \\ \hline \end{array}$$

3. $$\begin{array}{r} mr^2 \\ -mr^2 \\ \hline \end{array}$$

4. $$\begin{array}{r} \frac{1}{2}mv^2 \\ -\ \ mv^2 \\ \hline \end{array}$$

5. $$\begin{array}{r} -\ \ xy \\ 10xy \\ -15xy \\ 11xy \\ \hline \end{array}$$

6. $$\begin{array}{r} 4\pi r^2 \\ 2\pi r^2 \\ \hline \end{array}$$

7. $$\begin{array}{r} 12xyz \\ 13xyz \\ \hline \end{array}$$

8. $$\begin{array}{r} -\ \ mpq \\ -6mpq \\ \hline \end{array}$$

9. $$\begin{array}{r} -\frac{1}{2}mv^2 \\ -2mv^2 \\ \hline \end{array}$$

10. $$\begin{array}{r} 43w^3 \\ -23w^3 \\ 20w^3 \\ -10w^3 \\ \hline \end{array}$$

11. $$\begin{array}{r} -9pq^2 \\ -7pq^2 \\ \hline \end{array}$$

12. $$\begin{array}{r} \frac{1}{8}\pi r^3 \\ \frac{5}{8}\pi r^3 \\ \hline \end{array}$$

13. $$\begin{array}{r} -6a^3b \\ 8a^3b \\ \hline \end{array}$$

14. $$\begin{array}{r} 4xy^2 \\ -7xy^2 \\ \hline \end{array}$$

15. $$\begin{array}{r} 16uv^2 \\ -\ \ 9uv^2 \\ -\ \ 7uv^2 \\ 25uv^2 \\ \hline \end{array}$$

16. $$\begin{array}{l} 7x^2-2x-5 \\ 4x^2+4x+5 \\ \hline \end{array}$$

17. $$\begin{array}{l} 4.5m+3.2n+\ \ p \\ .5m \qquad\qquad\ +.1p \\ \hline \end{array}$$

18. $$\begin{array}{l} a^3+3a^2b+3ab^2+b^3 \\ a^3-3a^2b+3ab^2-b^3 \\ \hline \end{array}$$

19. $$\begin{array}{l} 3x-4y+7z-9 \\ 8y-2z+4x-3 \\ \hline \end{array}$$

WRITTEN EXERCISES

Add and test:

1. $a+b+c,\ 2a+b+3c,\ a+b,\ 6b+5c.$
2. $10a+9b+c,\ 9a+10c,\ \frac{1}{3}a+\frac{1}{4}b,\ 2b+c.$
3. $.9x+.3y+z,\ .1x+.7y,\ 5y+\frac{1}{2}z,\ 4x+z.$
4. $3a-2b,\ 4a+7b,\ -6a-b,\ 14a-21b.$
5. $m^2+5mp,\ 7m^2-8mp,\ -12m^2+3p^2,\ 6m^2-4mp+7p^2.$
6. $5ax+b,\ 2ax-3b,\ -8ax+7b,\ -20ax-18b.$
7. $\frac{1}{2}x-\frac{1}{3}y,\ -\frac{3}{5}x+\frac{1}{4}y,\ .7x+\frac{2}{3}y,\ -.3x+\frac{5}{6}y.$
8. $7+2x^2,\ 3x^2-1,\ 4-5x^2.$
9. $4a+7b,\ 2a-6c,\ 3c+5b,\ 4b-7a.$
10. $x^2+7x-4,\ 3x^2-5x,\ 4x^2-11x+2.$

11. $5t-3+7t^2,\ \ t-3t^3,\ \ t^3+9t^2-15+8t.$

12. $3ab+7ac,\ \ 5ac-2bc,\ \ 6bc+9ab,\ \ 8bc-18ac.$

13. $x+.4,\ \ x^2-.5,\ \ 3x-.7,\ \ x^2-.9+4x.$

14. $2y^2-4y+y^4-1,\ \ 8y-y^3+3y^2-15,\ \ 3y-7+11y^4-15y^3,\ \ 4y^3+12y^2-6+y^4,\ \ 11-y^4+y^3-8y^2.$

15. A grocer had 7 a dollars on hand; his ten salesmen took in 4 a, 5 c, 2 b, 6 a, 3 a, 7 c, 4 b, 2 c, 5 c, 11 b, dollars, respectively. How much had he then?

16. A merchant made the following bank deposits: On Monday 3 a dollars in gold, 4 b dollars in silver, and 9 c dollars in bills; on Tuesday a dollars in gold and 15 c dollars in bills; on Wednesday b dollars in silver and 12 c dollars in bills. How much did he deposit all together?

SUMMARY

The following questions summarize the definitions and processes of this chapter:

1. What is meant by *algebraic sum?*	Sec. 81.
2. What are *like terms* or *monomials?*	Sec. 82.
3. How may like terms be added?	Secs. 84, 85.
4. How may polynomials be added?	Sec. 87.
5. State the *Commutative Law of Addition.*	Sec. 88.
6. Explain a *test* for the work of addition.	Sec. 90.

CHAPTER VI

SUBTRACTION

SUBTRACTION OF MONOMIALS

91. Subtraction is the process of finding the difference between two numbers, called the *minuend* and the *subtrahend*.

92. Difference. The **difference** is the number which added to the subtrahend makes the minuend. Sec. 69.

ORAL EXERCISES

State the differences of the following:

1. $\begin{array}{r}20\,a\\ \underline{15\,a}\end{array}$

2. $\begin{array}{r}47\,b\\ \underline{27\,b}\end{array}$

3. $\begin{array}{r}12\,b^2c\\ \underline{7\,b^2c}\end{array}$

4. $\begin{array}{r}21\,x^2\\ \underline{14\,x^2}\end{array}$

5. $\begin{array}{r}17\,bc\\ \underline{9\,bc}\end{array}$

6. $\begin{array}{r}39\,c^2\\ \underline{19\,c^2}\end{array}$

7. $\begin{array}{r}90\,pq\\ \underline{45\,pq}\end{array}$

8. $\begin{array}{r}50\,zw\\ \underline{25\,zw}\end{array}$

9. $\begin{array}{r}30\,xy\\ \underline{17\,xy}\end{array}$

10. $\begin{array}{r}40\,mn\\ \underline{39\,mn}\end{array}$

11. $\begin{array}{r}54\,d\\ \underline{24\,d}\end{array}$

12. $\begin{array}{r}19\,abx\\ \underline{12\,abx}\end{array}$

93. Since subtraction is the reverse of addition, we can subtract a number by adding its opposite.

For example, "to subtract 3" and "to add -3" mean the same thing. Likewise "to subtract $-5\,a$" and "to add $5\,a$" mean the same thing.

To subtract one number from another, change the sign of the subtrahend and add the result to the minuend.

Thus, to subtract $\begin{array}{r}30\,a\\ \underline{-10\,a}\end{array}$ change to $\begin{array}{r}30\,a\\ \underline{10\,a}\end{array}$ and add.

The pupil should learn to make the change of sign mentally.

ORAL EXERCISES

Find the differences:

1. $\begin{array}{r}18\\ \underline{-99}\end{array}$

2. $\begin{array}{r}6ab\\ \underline{6ab}\end{array}$

3. $\begin{array}{r}5x\\ \underline{-15x}\end{array}$

4. $\begin{array}{r}5xy\\ \underline{-10xy}\end{array}$

5. $\begin{array}{r}-3abc\\ \underline{3abc}\end{array}$

6. $\begin{array}{r}-12m\\ \underline{8m}\end{array}$

7. $\begin{array}{r}12pq\\ \underline{-2pq}\end{array}$

8. $\begin{array}{r}6mv^2\\ \underline{19mv^2}\end{array}$

9. $\begin{array}{r}-1.5s\\ \underline{-3.5s}\end{array}$

10. $\begin{array}{r}-4\frac{1}{2}w\\ \underline{-9\frac{1}{2}w}\end{array}$

11. $\begin{array}{r}5a\\ \underline{8a}\end{array}$

12. $\begin{array}{r}4x\\ \underline{7x}\end{array}$

13. $\begin{array}{r}12y\\ \underline{3y}\end{array}$

14. $\begin{array}{r}-8x\\ \underline{2x}\end{array}$

15. $\begin{array}{r}-8x\\ \underline{-2x}\end{array}$

16. $\begin{array}{r}-14y\\ \underline{-6y}\end{array}$

17. $\begin{array}{r}16b\\ \underline{11b}\end{array}$

18. $\begin{array}{r}-14d\\ \underline{-7d}\end{array}$

19. $\begin{array}{r}8g\\ \underline{-3g}\end{array}$

20. $\begin{array}{r}-4p\\ \underline{-7p}\end{array}$

21. $\begin{array}{r}2m^2\\ \underline{-2m^2}\end{array}$

22. $\begin{array}{r}2m^2\\ \underline{2m^2}\end{array}$

23. $\begin{array}{r}23a\\ \underline{-6a}\end{array}$

24. $\begin{array}{r}23a\\ \underline{6a}\end{array}$

25. $\begin{array}{r}-23a\\ \underline{6a}\end{array}$

26. $\begin{array}{r}-23a\\ \underline{-6a}\end{array}$

27. $\begin{array}{r}23x^3\\ \underline{-6x^3}\end{array}$

28. $\begin{array}{r}-t\\ \underline{-3t}\end{array}$

29. $46 - 52.$

30. $4a - 7a.$

31. $8x^2 - 10x^2.$

32. $8ab - 15ab.$

33. $40m - 46m.$

34. $13pq - 15pq.$

35. $\frac{1}{2}mv^2 - mv^2.$

36. $\pi r^2 - 4\pi r^2.$

37. $\frac{4}{3}\pi r^3 - 2\pi r^3.$

38. $\frac{1}{2}gt^2 - gt^2.$

39. $10fs - 15fs.$

40. $45rt - 50rt.$

SUBTRACTION OF POLYNOMIALS

94. The subtraction of polynomials is similar to the subtraction of denominate numbers.

For example:

DENOMINATE NUMBERS	POLYNOMIALS
Just as: 5 lb. 4 oz.	so: $5l + 4z$
minus 3 lb. 3 oz.	minus $3l + 3z$
equals 2 lb. 1 oz.	equals $2l + 1z$

ORAL EXERCISES

Subtract:

1. $\begin{array}{r}\text{12 bu. 3 pk. 7 qt.}\\ \underline{\text{9 bu. 2 pk. 4 qt.}}\end{array}$

2. $\begin{array}{r}12b + 3p + 7q\\ \underline{9b + 2p + 4q}\end{array}$

State the numbers to fill the blanks in the following:

3. $$\begin{array}{r} 6xy+8y^2 \\ 3xy+2y^2 \\ \hline (\ )xy+(\ )y^2 \end{array}$$

4. $$\begin{array}{r} 5a+3b^2+8c \\ a+\ \ b^2+5c \\ \hline (\ )+(\ )+(\ ) \end{array}$$

Subtract:

5. $$\begin{array}{r} 6a+5b \\ a+3b \\ \hline \end{array}$$

6. $$\begin{array}{r} 16m+2n \\ 12m+\ \ n \\ \hline \end{array}$$

7. $$\begin{array}{r} 17c+9d \\ 17c+\ \ d \\ \hline \end{array}$$

8. $$\begin{array}{l} 20x+5y+z \\ \ \ 9x+5y \\ \hline \end{array}$$

9. $$\begin{array}{r} 45a+3b+12c \\ 5a+\ \ b+\ \ 2c \\ \hline \end{array}$$

10. $$\begin{array}{r} 10x^2+6y^2+5z^2 \\ 5x^2+\ \ y^2+5z^2 \\ \hline \end{array}$$

95. To subtract a polynomial *arrange its terms under the like terms of the minuend, subtract each term from the one above it, and use the signs obtained as the signs of the result.*

$$\begin{array}{r} 2a-2b+\ \ c \\ a+\ \ 0-2c \\ \hline a-2b+3c \end{array}$$

In the example, the first column is $2a-a=a$; the second column is $-2b-0=-2b$; the third column is $+c-(-2c)=+3c$. The terms thus obtained with their signs constitute $a-2b+3c$, the difference of the polynomials. When either polynomial lacks a term to correspond to a term of the other polynomial, supply zero in its place.

96. Test of Subtraction. Use arbitrary values to test subtraction. The sum of the values of the difference and the subtrahend must equal the value of the minuend.

SOLUTION

$$\begin{array}{r} 3x-4y+\ \ c \\ x+\ \ y-3c \\ \hline 2x-5y+4c \end{array}$$

TEST: Let each letter = 1

$$\begin{array}{r} 3-4+1=\ \ \ 0 \\ 1+1-3=-1 \\ \hline 2-5+4=+1 \end{array}$$

In practice it is sufficient to write the following:

$$\begin{array}{rc} \text{SOLUTION} & \text{TEST} \\ 3x-4y+\ \ c & 0 \\ x+\ \ y-3c & -1 \\ \hline 2x-5y+4c & 1 \end{array}$$

WRITTEN EXERCISES

Subtract and test:

1. $\begin{array}{r} 3a+b \\ \underline{a-b} \end{array}$

2. $\begin{array}{r} 6a-3b \\ \underline{5a+7b} \end{array}$

3. $\begin{array}{r} \frac{1}{2}a-\frac{1}{3}b \\ \underline{a+\frac{2}{3}b} \end{array}$

4. $\begin{array}{r} 3m-.1n \\ \underline{6m-.9n} \end{array}$

5. $\begin{array}{r} \frac{2}{3}p+\frac{1}{9}q \\ \underline{\frac{2}{3}p-\frac{8}{9}q} \end{array}$

6. $\begin{array}{r} -7x+4y \\ \underline{10x-9y} \end{array}$

Arrange the like terms in columns and subtract:

7. $\begin{array}{l} 8x+12q+6a \\ \underline{4a+\ 5x+3q} \end{array}$

8. $\begin{array}{l} p+3n+45m \\ \underline{5m+2n+p} \end{array}$

9. $\begin{array}{l} 5c+6b+10a \\ \underline{b+5a+\ 5c} \end{array}$

10. $\begin{array}{l} 10ax+\ mp+5pq \\ \underline{2pq+\frac{1}{2}mp+5ax} \end{array}$

11. $\begin{array}{r} 3xy-\ z^2 \\ \underline{5xy+10z^2} \end{array}$

12. $\begin{array}{r} 12t-\ 6t^2 \\ \underline{9t-12t^2} \end{array}$

13. $\begin{array}{r} 40-\frac{1}{2}gt^2 \\ \underline{50-\frac{1}{2}gt^2} \end{array}$

14. $\begin{array}{r} 5x+7y-8z \\ \underline{3x+\ y-4z} \end{array}$

15. $\begin{array}{r} 2a+c-2b \\ \underline{a+b-2c} \end{array}$

16. $\begin{array}{r} 7x^2-2x+4 \\ \underline{2x^2+3x-1} \end{array}$

17. $\begin{array}{r} 2z+4x+y \\ \underline{2x+\ y-z} \end{array}$

18. $\begin{array}{l} 6c+3d+e \\ \underline{3c+2d} \end{array}$

19. $\begin{array}{r} 3x^2+2x^2y^2+z^2 \\ \underline{x^2+2x^2y^2} \end{array}$

20. $\begin{array}{r} 4x-3y+8 \\ \underline{2y+5z-1} \end{array}$

21. $\begin{array}{r} 4a+2b-9 \\ \underline{8c+4a-6d} \end{array}$

22. $\begin{array}{r} 2x^4+5x-1 \\ \underline{3x^3-7x^2+8x} \end{array}$

23. A broker had $7a+5b$ dollars in a bank and withdrew $a+4b$ dollars. How much did he still have in the bank?

24. A coal dealer bought c carloads of coal containing 40 tons each, and d carloads of 50 tons each; he sold $5c$ tons to one customer, $8d$ tons to another, and $12c$ tons to another. How many tons had he left?

97. Removal of Parentheses.

1. If a parenthesis is preceded by the sign +, the terms within the parenthesis are to be added to what precedes, hence the parenthesis may be removed without altering the value of the expression.

For example:
$$a+(b+c)=a+b+c.$$
$$a+(b-c)=a+b-c.$$
$$a+(-b-c)=a-b-c.$$

2. If a parenthesis is preceded by the sign $-$, the terms within the parenthesis are to be subtracted from what precedes; hence the parenthesis may be removed provided the sign of each term within the parenthesis is changed, each sign $+$ to the sign $-$, and each sign $-$ to the sign $+$. Sec. 93.

For example:

$$a-(b+c)=a-b-c.$$
$$a-(b-c)=a-b+c.$$
$$a-(-b-c)=a+b+c.$$

WRITTEN EXERCISES

Remove parentheses and unite terms when possible:

1. $a+(b-3a)$.
2. $5-(t+2)$.
3. $3a-(2a+b)$.
4. $7b-(4a-6b)$.
5. $a-(2b-5a)-4b$.
6. $11t+(-3t-1)$.
7. $4x+7y-(3x+2y)$.
8. $d+3d^2-(2d-d^2)$.
9. $5-3p+(-18+2p)$.
10. $a-bx-(2a+bx)$.
11. $7q+5-(-11-3q)$.
12. $-(-3x-2y+11z)$.
13. $-(4a+3b-6c)$.
14. $x-3x^2+7-(2x^2+5-3x)$
15. $9t+3-(2t-1)$.
16. $5x-12y-(3x+2y)$.
17. $4a+7b-(2a+3b)$.
18. $7-6m-(1+3m)$.
19. $11p+1-(-p+3)$.
20. $15x^2+7x-(18x^2-3x)$.
21. $4m^3-7m^2+3m-(-2m^3-9m^2)$.
22. $a^2-5ab+7ac+b^2-(4ab+7a^2-6b^2)$.
23. $x^2+3xy-4xz+7y^2-(z^2-3xz-4x^2-y^2)$.
24. Calculate the value of $A-(2B-C)$ when

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
$A=$	24	1	-4	x	$4d$	$a+x$	$p+3$
$B=$	3	-5	3	$2x$	$-6d$	$3a-x$	$q+5$
$C=$	8	9	-5	$3x$	$3d$	$4a+7x$	$2p+3q$

98. The methods of Sec. 97 may be applied when there is a parenthesis within a parenthesis. In this case the signs, $\{\ \}$, $[\ ]$, $\overline{\quad}$, are commonly used to distinguish the different parentheses. They may be removed one at a time, usually the inner one first, although it is likewise possible to begin with the outer one.

For example:

Beginning with the inner parenthesis,

$$\begin{aligned} 5a-\{6a+3b-(2a-5b)\} &= 5a-\{6a+3b-2a+5b\} \\ &= 5a-\{4a+8b\} \\ &= 5a-4a-8b \\ &= a-8b. \end{aligned}$$

Beginning with the outer parenthesis,

$$\begin{aligned} 5a-\{6a+3b-(2a-5b)\} &= 5a-6a-3b+(2a-5b) \\ &= -a-3b+2a-5b \\ &= a-8b. \end{aligned}$$

When the first parenthesis is removed the signs within the second one are not changed because the expression is taken as a single term.

WRITTEN EXERCISES

Remove parentheses and unite terms as much as possible:

1. $4x-\{3x-(2+x)\}$.
2. $7+\{4-(5x+2)+3x\}$.
3. $2+(5a-\overline{3a+4a})+6a$.
4. $x^2-\{3x^2-(2x^2+1)\}$.
5. $-\{4x^2+(3x-\overline{5x^2-9x})\}$.
6. $a-3b-\{b-3a+(3b-a)\}$.
7. $p+(3p-q)-(3q-p)+q$.
8. $a^2-(b^2-c^2+a^2)-(c^2-b^2)$.
9. $3xy+2y^2-(x^2-xy+2y^2)$.
10. $3a+b-\{c-[d-\overline{3a+b}-c]+d\}$.
11. $m-\{3m-[4m-(5m-\overline{6m-7m})-8m]+9m\}+10m$.
12. $5a-[2x-\{4c+(7c-x)-3c\}+4a]$.
13. $a^4-[a^2-(1-a)]-[1+\{a^2-(1-a)+a^4\}]$.
14. $6m-\{p+2q-(m+q)+3-(5p-3q-4m)\}$.
15. $ab-ac-\{2ab-(3ac+bc)-2bc+2ac-(ab-2ac)\}$.
16. $b-[x-\{3+3b-(5x+2)\}-(4x-3b-1)]$.

17. $m+p-\{-[2m+2p-(q+r)]-[4p$
$-(r-q+m)-p]+3m\}$.

18. $2a+b+(12a-4b)-\{3a+3b-(5a-\overline{3a-3b})\}$.

19. $\frac{1}{3}a-\frac{1}{2}x-(\frac{3}{4}a-\frac{1}{6}x)-(2n-\overline{\frac{5}{4}x-\frac{1}{4}a})+\frac{5}{6}a$.

20. $3-\{[4y-(3-\overline{a-2})]-[a+(5y-\overline{a+5})]\}$.

21. $xy-[3+a-(x+z-xy+a)]+\{a-(y-z-5)\}$.

22. $-[3ab-\{5bc-3d\}+d-(4bc+\{2d+3ab\}+d)]$.

23. $x-(y-z)-\{x-[y-z-(y+z-x)+(x-y)]\}$.

24. $1-y-\{1-y-[1-y-(1-y)-(y-1)]\}$.

25. $.5a-\{a+\overline{.3ab-.3a}-[.25a+\overline{c+1.5a}]\}$.

99. Introduction of Parentheses.

The value of a polynomial is not changed:

1. If any number of terms with their *signs unchanged* are grouped in a parenthesis preceded by the *sign* +.

2. If any number of terms with their *signs changed* are grouped in a parenthesis preceded by the *sign* −. Sec. 97.

For example:

$$2a+5b-6c=2a+(5b-6c).$$
$$2a+5b-6c=2a-(-5b+6c).$$
$$4a-7m+2x=4a-(7m-2x).$$

100. The fact that the terms of a polynomial may be grouped as stated in Sec. 99, without changing the value of the polynomial, is called the **Associative Law of Addition.**

WRITTEN EXERCISES

Write as a plus a parenthesis:

1. $a+6b-4c$. 3. $a-4b-c+5d$. 5. $a+9b-7c+3d$.
2. $a-3b+7c$. 4. $a-8b+c-1$. 6. $a+2b+7c+4$.

7–12. Write each of the expressions in Exercises 1–6 as a minus a parenthesis.

13–18. In each of the expressions in Exercises 1–6 place the terms involving b and c in a parenthesis preceded by the sign −.

Group the like terms in a parenthesis preceded by a minus sign:

19. $bc - a^2 + d + 3a^2 + c - 5a^2$.

20. $x^2 - 2xy + y^2 - 3xy + z$.

21. $ab + bc - 2ab - cd + 5ab$.

22. Collect in a parenthesis the like terms in x in $y^2 - 2x + x^2 - px + z^2 - pqx$.

23. Collect the like terms in x^2, also those in y, each in a parenthesis preceded by a minus sign:

$$x^2 - y^2 - px^2 + 2y + 5x^2 - xy + py.$$

24. Taking the terms of this expression in order, group each pair in a parenthesis:

$$3a^2 - b - c^2 - d^2 + bc + ac - x^2 - xyz.$$

101. The sum or the difference of terms alike with respect to certain letters may be indicated by the use of parentheses.

EXAMPLES

1. Add ax and bx.

$$\begin{array}{lr} \text{Addend} & ax \\ \text{Addend} & bx \\ \hline \text{Sum} & (a+b)x \end{array}$$

2. Subtract $(c+n)xy$ from $(b+2c)xy$.

$$\begin{array}{r} (b+2c)xy \\ (c+\ \ n)xy \\ \hline (b+c-n)xy \end{array}$$

WRITTEN EXERCISES

Add:

1. $\begin{array}{r} (a+b)x \\ bx \\ \hline \end{array}$

3. $\begin{array}{r} (a+c)xy \\ (a-c)xy \\ \hline \end{array}$

5. $\begin{array}{r} (a+c)(m^2+n) \\ c(m^2+n) \\ \hline \end{array}$

2. $\begin{array}{r} (1+2b)x^2 \\ -bx^2 \\ \hline \end{array}$

4. $\begin{array}{r} (a+b)(p+q) \\ (c+d)(p+q) \\ \hline \end{array}$

6. $\begin{array}{r} (m-14+p)x^2y \\ (-m+2n-p)x^2y \\ \hline \end{array}$

7. Collect the terms in x and put the coefficients in a parenthesis with the proper sign:

$$a - x + bx - abx.$$

8. Similarly, collect the terms in x, also those in y:

$$ax - by - bx - ay.$$

9. Similarly, collect the like terms of each kind:

$$3a - 2b - 6a - c - 5b + 10c.$$

10. Similarly, collect the like terms in x^2, and also in xy:

$$2x^2 - xy + ax^2 - 3xy - bx + 2axy.$$

Group in parentheses the same powers of x in each:

11. $ax^2 - bx^3 + cx - ax^3 + bx^2 - dx.$

12. $x^2 - 3x^3 + ax^2 - 2x + cx^3 + bx.$

13. $mx^3 + 2px + qx^2 - qx - px^3 - px^4.$

14. $ax^3 - x^4 + bx^2 - ax^4 + bx^3 - px.$

Collect in alphabetical order the coefficients of x, also of y:

15. $2ax - by + 3bx - 4ay - cx + 2cy.$

16. $mx + py - qx + ny - 5px + 4nx.$

Subtract:

17. $$\begin{array}{r}(a+2b)x^p\\ bx^p\\ \hline\end{array}$$

19. $$\begin{array}{r}(a+c)(m^2+n)\\ (2a+c)(m^2+n)\\ \hline\end{array}$$

21. $$\begin{array}{r}(a+c)(x+y).\\ 2c(x+y)\\ \hline\end{array}$$

18. $$\begin{array}{r}(2m-4)x^2y\\ (37-n)x^2y\\ \hline\end{array}$$

20. $$\begin{array}{r}(a+c)x^p\\ (a+2c)x^p\\ \hline\end{array}$$

22. $$\begin{array}{r}(1+b+c)p^2q^2\\ (a-2b-2)p^2q^2\\ \hline\end{array}$$

First collect like terms, then subtract:

23. $ax + bx$ from $cx + dx$.

24. $ax - bx$ from $cx - dx$.

25. $ax - 3bx$ from $ax - 4bx$.

26. $ax + 4bx$ from $2ax - bx + cx$.

27. $ax + 2y - 3x$ from $ax - y + 2x$.

28. From $(a+b)x^2$ plus $(c-a)x^2$ subtract $(b-c)x^2$.

29. From $(a-2b)xy$ plus $(b-2a)xy$ subtract $(a+b+c)xy$.

30. From $(a+b)x + (c+d)y$ plus $2bx - 2cy$ take $2ax - (2d-c)y$.

REVIEW

ORAL EXERCISES

Subtract:

1. $\begin{array}{r} 5n \\ \underline{16n} \end{array}$

2. $\begin{array}{r} 7a \\ \underline{-9a} \end{array}$

3. $\begin{array}{r} 7a-5b \\ \underline{a-2b} \end{array}$

4. $\begin{array}{r} x+y \\ \underline{x-x} \end{array}$

5. $\begin{array}{r} a-7 \\ \underline{2a-3} \end{array}$

6. $\begin{array}{r} x^2+y \\ \underline{2x^2-y} \end{array}$

7. $\begin{array}{r} 11a+10b \\ \underline{a+3b} \end{array}$

8. $\begin{array}{r} 3a-b \\ \underline{3a+b} \end{array}$

9. $\begin{array}{r} cx-dy \\ \underline{x-y} \end{array}$

10. $\begin{array}{r} a-x \\ \underline{a-1} \end{array}$

11. $\begin{array}{r} p^2-q^2 \\ \underline{p^2+q^2} \end{array}$

12. $\begin{array}{r} ax-b \\ \underline{bx-c} \end{array}$

13. $\begin{array}{r} (a-2b)xy \\ \underline{(b-a)xy} \end{array}$

14. $\begin{array}{r} (2a+b-c)x^2 \\ \underline{(a-2b-c)x^2} \end{array}$

15. $\begin{array}{r} (a+2+2c)y^{2n} \\ \underline{(1+a)y^{2n}} \end{array}$

WRITTEN EXERCISES

Remove the parentheses:

1. $(a+b-c)+(a-b+c)$.
2. $(a+b-c)-(a-b+c)$.
3. $7a-3b-(5a+3b)$.
4. $3x-7-(9x-11)$.
5. $2y-(\frac{1}{2}tx+\frac{1}{8}y)$.
6. $(\frac{1}{3}tx-\frac{1}{2}y)-(\frac{1}{6}y-\frac{1}{6}tx)$.
7. $m-[(a-b)-(c-m)]$.
8. $m+[(a-b)+(b+d)]$.
9. $6x+5y-32-(5x-3y+22)$.
10. $6a^2-(3ab+2ac)-(2ac+3ab)$.

Subtract and test:

11. $\begin{array}{r} 3x^2+2x^2y^2+z^2 \\ \underline{x^2-2x^2y^2} \end{array}$

12. $\begin{array}{r} m-3n+p-7 \\ \underline{m-4n-p+8} \end{array}$

13. $\begin{array}{r} 6c+3d+a-3z \\ \underline{3c-2d-2a} \end{array}$

14. $\begin{array}{r} 4x+3y-4z+8 \\ \underline{-7x-3y-2z+17} \end{array}$

15. $\begin{array}{l} 8x^3-9x^2+x^4-x+16+x^5 \\ \underline{2x-7+5x^4-x^2-x^3+6x^5} \end{array}$

16. $\begin{array}{r} 5.65a+7\frac{3}{5}b-27\frac{3}{4}c+.76x-1\frac{1}{8}y \\ \underline{4\frac{1}{4}a-9.38b+2.65c-13\frac{1}{2}x-0.375y} \end{array}$

17. Group the last four terms in a bracket and the last three in a parenthesis:

$$mp - 3\,mn + 2\,pq + m^2 - n^2.$$

18. Group as the difference of two trinomials without changing the order of the terms:

$$x^2 + 2\,xy + y^2 - x^2 + 2\,xy - y^2.$$

Test by removing the parentheses and point out any errors:

19. $a^3 - a^2 + a - 1 = (a^3 - 1) - (a^2 + a).$

20. $a^2 - b^2 + 2\,bc - c^2 = a^2 - (b^2 + 2\,bc - c^2).$

21. From $(2\,b + c)x$ take $(b - c)x$.

22. From $(a + 2\,b - c)xy$ take $(2\,a - b + 2\,c)xy$.

23. From the sum of $(2\,a + b)x^2$ and $(c - b)x^2$ take $(a + b - c)x^2$.

24. Collect into the first member the terms in x and unite:

$$15 - (2\,a + b)x + 3\,cx = 20 + (c - 2\,b)x - 4\,ax.$$

SUMMARY

The following questions summarize the definitions and processes in this chapter:

1. Define *subtraction;* also *difference.* Secs. 91, 92.
2. Explain how to *subtract monomials.* Sec. 93.
3. Explain how to *subtract polynomials.* Sec. 95.
4. Explain the test of subtraction by arbitrary values. Sec. 96.
5. State the sign rules for removing parentheses. Sec. 97.
6. State the sign rules for introducing parentheses. Sec. 99.

CHAPTER VII

EQUATIONS

102. Practice in writing expressions containing positive and negative numbers is a necessary preparation for solving equations involving these numbers.

EXAMPLE

By how much does $b + 3$ exceed $b - 3$?

Indicating the difference and simplifying:

$$b + 3 - (b - 3) = b + 3 - b + 3 = b - b + 3 + 3 = 6.$$

WRITTEN EXERCISES

By how much does

1. $2x - 5$ exceed $x - 5$?
2. $2a - x$ exceed $2a + x$?
3. $2x - 5$ exceed $x - 6$?
4. $4a - 1$ exceed $2a - 20$?
5. B's age is $3x - 1$ years. What was it 5 years ago? What will it be $x + 2$ years hence?

If A is $n + 2$ years old and B is $2n - 1$ years old, express:

6. A's age 4 years ago.
7. B's age 4 years ago.
8. A's age 5 years hence.
9. B's age 5 years hence.
10. The sum of their ages now.
11. The sum of their ages 4 years ago.
12. The sum of their ages 5 years hence.
13. How much older is B than A?

If A has $d + 25$ dollars and B has $2d - 6$ dollars, write the equation expressing that:

14. A and B together have $100.
15. A has $20 more than B.
16. B has $20 less than A.

17. B's money increased by \$15 equals A's money decreased by \$10.

18. If A loses \$10 and B gains \$10, they will have the same amounts.

19. Express the next larger integer after n. Also the second one. Also the third.

20. Express the next integer smaller than n; the second integer smaller than n; the third.

Express:

21. The sum of two consecutive integers is 27.

22. The sum of three consecutive integers is 15.

23. The next larger integer after $n-5$; after $n-1$.

24. The next integer smaller than $n-5$; than $n-1$.

25. The sum of $n-4$ and the two next smaller integers.

103. Certain classes of numbers are represented by algebraic formulas.

EXAMPLES

1. $2n$ represents all even numbers, if n be given all positive integral values.

2. $2n-1$ represents all odd numbers, if n be given all positive integral values.

3. $n+\frac{m}{10}+\frac{p}{100}$ represents any number with two decimal places, where n is the whole number and m and p are the decimal figures.

4. $\frac{n}{m}$ represents any fraction. If n and m are integers and $n<m$, it is a proper fraction.

5. n^2 represents square numbers. It represents the squares of all integers if n has all positive or all negative integral values.

WRITTEN EXERCISES

1. In $2n$ substitute for n the numbers that will produce the even numbers between 99 and 111.

2. In $2n$ substitute for n the numbers that will produce the multiples of 4 between 15 and 25.

3. If the negative integers be substituted for n in $2n$, what kind of numbers are produced?

4. When n is 1, what is the value of $2n+1$? When n is 2? When n is 3? 4? 0? What class of numbers does $2n+1$ represent when n may be zero or any positive integer?

5. Write a formula to represent all positive cube numbers. Can the values of the letter be negative? If they are taken as negative, what class of numbers is produced?

6. Can you substitute in n^2 any numbers that will make the series, -1, -4, -9, -16, and so on?

7. What number does $n+\frac{m}{10}+\frac{p}{100}$ become when $n=1$, $m=2$, $p=3$? Also when $n=63$, $m=0$, $p=8$? Also when $n=0$, $m=2$, $p=5$?

8. Write a formula to represent any number with three decimal places.

9. If q is 10 in $\frac{p}{q}$, what is the greatest integral value p can have and make $\frac{p}{q}$ a proper fraction?

10. What kind of a fraction is $\frac{n}{n+1}$ for all positive integral values of n?

104. The processes with negative numbers may be applied in substitution.

EXAMPLES

1. What is the value of d in $d=vt$, when $v=-3$ and $t=8$?

Given $\qquad d = vt.$ (*1*)

When $v=-8$ and $t=8$, $\qquad d=-3\cdot 8.$ (*2*)

Therefore, $\qquad d=-24.$ (Sec. 75.) (*3*)

2. What is the value of t in $d=vt$, when $d=-16$, $v=-2$?

Given $\qquad d = vt.$ (*1*)

Dividing by v, $\qquad t=\frac{d}{v}.$ (*2*)

When $d=-16$ and $v=-2$, $\qquad t=\frac{-16}{-2}.$ (*3*)

Therefore, $\qquad t=8.$ (Sec. 77.) (*4*)

3. Find a in $a = \pi r^2$, when $r = -2$.

When $r = -2$, $a = \pi(-2)^2$. *(1)*

Therefore, $a = 3.1416 \times (-2)(-2)$. (Sec. 75.) *(2)*

$= 12.5664$.

ORAL EXERCISES

1. In $w = fs$, find w when $f = 6$ and $s = -3$. Also find w when $f = 12$ and $s = -\frac{1}{8}$.

2. In $w = fs$, find f when $w = 20$ and $s = -5$.

3. In $d = vt$, find d when $v = -12$ and $t = \frac{1}{4}$. Also find t when $d = -2$ and $v = -\frac{1}{8}$.

4. In $s = \frac{1}{2}ab$, find s when $a = 6$ and $b = -5$. Also find s when $a = -8$ and $b = -3$.

5. In $x = 2ay$, find x when $y = -15$. Also find y when $x = -10$.

6. In $3x = az$, find x when $z = -9$. Also find z when $x = 4a$.

WRITTEN EXERCISES

1. In $v = \pi r^2 a$, find v when $\pi = 3.1416$ and $r = -25$.

2. In $v = \pi r^2 a$, find a when $v = 9.4248$ and $r = -\frac{1}{2}$.

3. In $k = \frac{1}{2}mv^2$, find k when $m = -12$ and $v = 3$. Also find k when $m = -15$ and $v = -2$. Also find k when $m = 20$ and $v = -5$.

4. In $a = \pi r^2$, find a when $r = -4$; also when $r = -\frac{1}{8}$.

5. In $c = 2\pi r$, find c when $r = 20$ ft.; also when $r = -15$ ft.

6. In $s = \frac{1}{2}ab$, find s when $a = -17$ and $b = -12$; also when $a = 15$ and $b = -9$.

7. In $s = \frac{1}{2}ab$, find a when $s = -42$ and $b = -7$.

8. In $x = 3y^2$, find x when $y = 14$; also when $y = -14$.

9. In $3x + 1 = 4y - 4$, find x when $y = -5$.

10. In $ax = by + bc + a$, find x when $y = -c$.

11. In $ay - b = bx + c$ find x when $y = \frac{c}{a}$. Also find y when $x = -1$.

105. The processes with negative numbers may be applied in solving equations.

EXAMPLES

1. Solve for x: $3x - 4 = 11$ *(1)*

Add 4 to both members, $4 = 4$

Then, $3x = 11 + 4 = 15$, (Sec. 67.) *(2)*

and $x = 5$. *(3)*

TEST. $3 \cdot 5 - 4 = 11$.

2. Solve for x: $5x + 21 = 6$ *(1)*

Subtract 21 from both members, $21 = 21$

Then, $5x = 6 - 21 = -15$, (Sec. 71.) *(2)*

and $x = -3$. (Sec. 77.) *(3)*

TEST. $5(-3) + 21 = 6$.

The effect of adding or subtracting as above is equivalent to taking a number from one side of the equation and placing it on the other with the sign changed, and the term **transpose** is often used to name this process; compare steps (1) and (2) in each example. But it is much better at first in explaining the work to say that a certain number has been added to, or subtracted from, both members of the equation.

3. Solve for y: $2y - 6 = 3y + 1$. *(1)*

Subtracting $3y$ from both members, $-y - 6 = 1$. (Sec. 71.) *(2)*

Adding 6 to both members, $-y = 7$. (Sec. 67.) *(3)*

Multiplying both members by -1, $y = -7$. (Sec. 75.) *(4)*

TEST. $2(-7) - 6 = 3(-7) + 1$.

The change from step (3) to step (4) follows directly from the meaning of relative numbers. That is, if negative y equals positive 7, then positive y must equal negative 7. But since the sign of a number is changed by multiplying or dividing it by -1, it is customary to refer such a change of signs to one or the other of these processes.

ORAL EXERCISES

Solve:

1. $x + 1 = -8$.
2. $x - 2 = -5$.
3. $2x - 2 = 6$.
4. $2y - 1 = 11$.
5. $3y - 5 = 13$.
6. $7x + 5 = -9$.
7. $2 - 3x = -4$.
8. $5 - 4x = 21$.
9. $\frac{1}{2}y - 4 = 8$.
10. $5y + 5 = -20$.
11. $x - a = b - c$.
12. $ax - 2b = c - 3b$.
13. $4c - 2bx = 8 - 4bx$.
14. $y - a = -b$.
15. $ay + b = -c$.

WRITTEN EXERCISES

Solve for x:

1. $2x-1=3x-7.$
2. $5x+3=2x-15.$
3. $4x-5=10-x.$
4. $\frac{1}{2}x+6=x-18.$
5. $\frac{3}{2}x-5=20-x.$
6. $x+7=2-\frac{1}{4}x.$
7. $\frac{2}{3}-3x=4x-\frac{1}{3}.$
8. $\frac{5}{6}x-8=-17-\frac{1}{6}x.$
9. $ax-b=\frac{1}{2}ax+c.$
10. $ax+m=bc-(mx+m).$

Solve for the letter in each case:

11. $x-3+4x+21-2x=0.$
12. $y+8+6y-4-3y=0.$
13. $2y-5-y+20=-15.$
14. $m-7+3m+42=-4m.$
15. $5t+6-3t-10=t.$
16. $12-h+8-4h=-10.$
17. $8-2p+6-5p=21.$
18. $7s-16=3-3s-29.$
19. $35-4y=6y+5+5y.$
20. $2+x-5=8x+11.$
21. $10+z-4=-5z-12.$
22. $8-8w=28-28w.$

106. The process of removing parentheses applies to equations.

EXAMPLE

Solve: $x-(2x+6)=3-(4x-6).$ (*1*)

Removing the parentheses, $x-2x-6=3-4x+6.$ (Sec. 97.) (*2*)

Then, $3x=15,$

and $x=5.$

TEST. $5-(2\cdot 5+6)=-11=3-(20-6).$

WRITTEN EXERCISES

Solve and test:

1. $2x-(2+x)=6.$
2. $\frac{1}{2}x+(2-x)=6.$
3. $4x-(2x+1)=5.$
4. $5-(4x+2)=4.$
5. $7-(4x+3)=0.$
6. $6y+(4y-10)=10.$
7. $12z-(2+6z)=16.$
8. $2\frac{1}{2}+(2\frac{1}{2}-5t)=0.$
9. $2y-(y+6)=2y-7.$
10. $3x-(2x+8)=2-4x.$
11. $7+t=3-(2t-17).$
12. $5-3s=6-(s+15).$
13. $z-(6-5z-18)=30.$
14. $4p-(3p+1)=2p+1.$
15. $5w-(w+6)=2-(w+1).$
16. $z-6=z-(3z+17).$

17. $-(x+5)=8+(x-5)$. 19. $2y-(y+5)=4-(y-3)$.
18. $15-(z+12)=4z-20$. 20. $16-(2x+8)=3x+(4x-1)$.

107. In solving problems by means of equations, the parenthesis is often used.

EXAMPLE

The sum of two numbers is 100, and 3 times one of them is 7 times the other; what are the numbers?

Solution. 1. Let x be the smaller number.
2. Then, $100-x$ is the larger number.
3. Then, $7x$ is 7 times the smaller one and $3(100-x)$ is 3 times the larger one.
4. $\therefore 7x=3(100-x)$, according to the problem.
5. $\therefore 7x=300-3x$. (Sec. 24.)
6. $\therefore 10x=300$.
7. $\therefore x=30$, and the numbers are 30, 70.

Test. $30+70=100$, and $7\times 30=3\times 70$. Therefore the numbers found fulfill the conditions of the problem.

The use of the parenthesis is seen in the third and fourth steps.

WRITTEN EXERCISES

1. Write an equation stating that if the cost (c) of a lot be diminished by $\$200$ and the remainder multiplied by 5, the result will be the value (v) of the house.

2. Write the equation which states that c times the sum of x and a equals d.

3. A father is twice as old as his son; 11 years ago he was three times as old as his son. Find the age of each.

4. If a certain number is diminished by 9 and the remainder multiplied by 9, the result is the same as if the number were diminished by 6, and the remainder multiplied by 6. Find the number.

5. Divide 48 into two such parts that one part shall exceed the other by 6.

6. A man is four times as old as his son; in 18 years he will be only twice as old. Find the age of each.

7. Two men enter a partnership and together furnish a capital of \$5000; twice what one furnishes is 3 times what the other furnishes. How much does each furnish?

8. The sum of two numbers is 40; the smaller number, x, is $\frac{1}{3}$ of the larger number. Write the equation needed to find x. Find the numbers.

9. A rectangular lot is 20 ft. longer than it is wide. Using x to represent the width, state what represents the length. Write an equation stating that 4 times the width equals 2 times the length. Find the dimensions of the lot.

10. In a certain post office there are three rates of pay: \$50, \$100, and \$150 per month. There are 5 more men receiving \$100 than \$150, and 2 more receiving \$50 than \$100; the monthly pay roll is \$1150. Letting x represent the number receiving \$150, write the equation needed to find x.

108. The solution of problems must be proved by substituting the results in the conditions of the problems.

EXAMPLE

1. A commission merchant remitted \$475 as the proceeds of a sale of 500 bu. of potatoes after deducting his commission of 5%. For how much did he sell the potatoes?

SOLUTION. 1. Let $x =$ the number of dollars received for the potatoes.
2. Then, $x - .05x =$ the amount remitted.
3. $\therefore x - .05x = 475$, according to the problem.
4. $\therefore .95x = 475$.
5. Dividing by .95, $x = 500$.
6. $\therefore$ the potatoes sold for \$500.

TEST. $500 - .05 \cdot 500 = 475$.

This merely tests the correctness of the work after step 3; it does not test the correctness of the equation in step 3; to do this the result, 500, must be tested *in the conditions of the problem itself.*

Thus, (1) 5% of \$500 = \$25, (2) \$500 − \$25 = \$475, the proceeds. If equation 3 had been incorrectly written, the result found in step 6 might have been a correct solution of equation 3 without giving the correct proceeds.

WRITTEN EXERCISES

Solve and test:

1. If A has x dollars and B has twice as much, express what they both have. If this amount is \$120, write and solve an equation, thus finding what each has.

✓ 2. The sum of a number and .05 of itself is 210, what is the number?

3. Separate the number 72 into two such parts that one part shall be $\frac{1}{8}$ of the other.

4. Divide 100 into two parts such that one is 24 less than the other.

5. Three times a number less twice the difference between the number and 5 is 30. Find the number.

6. Find a number such that when 9 is added to three times the number the sum is 42.

7. A tree 120 ft. high was broken so that the length of the part broken off was four times the length of the part left standing. Find the length of each part.

8. Three men together have \$1800. The first has twice as much as the second, and the third has as much as the first and second. How much has each?

✓ 9. A man was hired for 50 da. Each day he worked he was to receive \$2 and each day he was idle he was to forfeit 50¢ instead of getting his wages. At the end of the 50 da. his balance was \$80. How many days did he work?

10. A farmer bought 16 sheep. If he had bought 4 sheep more for the same money, each sheep would have cost him one dollar less. How much did he pay for a sheep?

11. A, B, and C bought a summer cottage for \$3000. B pays twice as much as A, and C pays as much as A and B together. How much does each pay?

12. A tree 90 ft. high was broken so that the length of the ✓ part broken off was 5 times the length of the part left standing Find the length of each part.

HISTORICAL NOTE

The earliest attempts to solve equations mark the beginning of algebra. About 1700 B.C. there lived an Egyptian priest named Ahmes. He was probably a teacher of mathematics, and was the first writer on the subject whose work is still in existence, for there has been recovered from the ancient pyramids a manual written by him which contains, among other mathematical subjects, a simple treatment of equations. Ahmes' work, called *Directions for obtaining the Knowledge of all Dark Things*, is written in hieroglyphics on papyrus, and is now, thirty-six centuries after its production, the property of the British Museum. It was deciphered by the German scholar, Eisenlohr, in 1877.

The equations of Ahmes were expressed in words, not in symbols; for example, one of his problems, when translated, reads: "Hau (literally *heap*, meaning unknown quantity) its seventh, its whole, makes nineteen," which in present-day algebraic notation is $x + \frac{x}{7} = 19$. There are few rules in Ahmes' handbook to show how his results were obtained, but it contains certain tables of numbers that indicate the processes. For example, he could write only unit-fractions, like $\frac{1}{2}$, $\frac{1}{5}$, and $\frac{1}{7}$, other fractions being expressed by the sum of unit-fractions. Accordingly, he gives the solution of the above equation as follows:

(1) $8 \quad \frac{1}{7}x = 19$ (2) $\frac{1}{7}x = 2 \quad \frac{1}{4} \quad \frac{1}{8}$ (3) $x = 16 \quad \frac{1}{2} \quad \frac{1}{8}$

(In the first step the adjacent numbers are supposed to be multiplied, and in the other steps they are to be added.)

Ahmes was greatly hindered by lack of suitable notation and other limitations, and knowing nothing of negative number, he could not solve so simple an equation as $x + 3 = 1$.

Little progress was made in solving equations during the twenty centuries from Ahmes to Diophantos. During the Middle Ages the Arabs and Hindoos simplified the solutions of Diophantos, but the perfection of modern symbols and the explanation of positive and negative roots remained for the mathematicians of the sixteenth century.

CHAPTER VIII

MULTIPLICATION

MULTIPLICATION OF MONOMIALS

109. PREPARATORY.

1. $5 \cdot 3¢ = (\ )¢.$
2. $6 \cdot 3d = (\ )d.$
3. $5 \cdot 3$ ft. $= (\ )$ ft.
4. $7 \cdot 5x = (\ )x.$
5. $5 \cdot 3f = (\ )f.$
6. $6 \cdot 6abc = (\ )abc.$
7. $3a \cdot 4b = 3 \cdot 4ab = ?$
8. $\frac{1}{2}a \cdot 8b = \frac{1}{2}$ of $8ab = ?$
9. $4x \cdot 6y \cdot \frac{2}{3}z = 4 \cdot 6 \cdot \frac{2}{3}xyz = (\ )xyz.$

110. To find the product of two monomials, *take the product of their literal parts for the literal part of the product, and the product of their coefficients for the coefficient of the product.*

Prefix to the result the sign $+$ *or* $-$ *according to Sec. 75.*

111. Repeated Factors. If the same letter occurs more than once as a factor in the product, it should be written only once, with the proper exponent (Sec. 14).

For example:

$$-3a \cdot 2ab = -6aab = -6a^2b.$$
$$a^2b^3 \cdot ab^2 = aabbb \cdot abb = aaa \cdot bbbbb = a^3b^5.$$

ORAL EXERCISES

State the products:

1. $t^2 \cdot t^4.$
2. $y^5 \cdot y^2.$
3. $x^2y \cdot y^3.$
4. $8c \cdot 8c.$
5. $7a \cdot 6ab.$
6. $a^3 \cdot -a^2.$
7. $4a \cdot 2ax.$
8. $4x^3 \cdot -2x^2.$
9. $a^2x \cdot -a^3x^4.$
10. $12ab \cdot 2ab^2.$
11. $3xy^2 \cdot -5x^2y.$
12. $2a^4 \cdot a^5.$
13. $(4p^2)(8p^4).$
14. $(am^3)(-am^3).$
15. $2a^3r \cdot 6a^2r^5.$
16. $7a^2 \cdot -a^2.$
17. $15v^3 \cdot v^5.$
18. $13x^7 \cdot x^3.$

WRITTEN EXERCISES

Multiply as indicated:

1. $48\,ab^3 \cdot -10\,bc^2$.	4. $16\,a^2 \cdot 15\,a^3c$.	7. $30\,x^2 \cdot -4\,ax^3$.
2. $24\,a^2bc \cdot 12\,c$.	5. $36\,x^2 \cdot 3\,xz^2$.	8. $65\,a^2 \cdot -4\,a^9b$.
3. $214\,bc \cdot 6\,b^2c^3$.	6. $12\,m^5 \cdot -16\,mr^3$.	9. $27\,a^6 \cdot -63\,a^9b^2$.

112. Law of Exponents in Multiplication. *The exponent of any letter in a product is the sum of the exponents of that letter in all of the factors.*

In symbols this law is expressed:

$$a^m \cdot a^r = a^{m+r}.$$

EXAMPLES

1. $a^5 \cdot a^7 = a^{5+7} = a^{12}$.
2. $x^n \cdot x^{n+2} = x^{n+n+2} = x^{2n+2}$.
3. $$\begin{aligned} a^n \cdot -a^7x \cdot ax^{p-3} &= -a^n \cdot a^7 \cdot a^1 \cdot x^1 \cdot x^{p-3} \\ &= -a^{n+7+1} \cdot x^{1+p-3} \\ &= -a^{n+8}x^{p-2}. \end{aligned}$$

113. To find the product of several monomials:

1. *Find the sign of the product.*

The sign is plus when the number of negative factors is even, and minus when the number of negative factors is odd.

2. *Find the numerical coefficient of the product.*
3. *Find the literal part of the product.*

EXAMPLE

Find the product of $-2\,a$, $+3\,b$, $-2\,ab$, $-4\,bc$.

1. The sign is $-$.
2. $2 \cdot 3 \cdot 2 \cdot 4 = 48$.
3. $a \cdot b \cdot ab \cdot bc = a^2b^3c$.
4. $\therefore -2\,a \,.\, 3\,b \cdot -2\,ab \cdot -4\,bc = -48\,a^2b^3c$.

114. *If any factor is zero, the product is zero.*

For example:

$$3 \cdot 0 = 0;\ \text{similarly,}\ -x \cdot 0 = 0,\ \text{and}\ x \cdot y \cdot z \cdot 0 = 0.$$

This follows from the meaning of zero Sec. 66. For if $3 - 3 = 0$, then $(3 - 3)4$ represents $0 \cdot 4$; but $(3-3)4 = 12 - 12 = 0$.

Hence, $0 \cdot 4$ should be taken to mean 0.

ORAL EXERCISES

Multiply:

1. $6, -3, 5.$
2. $-7, 3, -1.$
3. $a, 0, -c.$
4. $-6, -0, -4.$
5. $2x, -3y, -2z.$
6. $ax, -bx, -cx.$
7. $-a, -a, 0.$
8. $(-q), (-q), (-q).$
9. $(-6), (-6).$
10. $(-4x), (-4x).$
11. $(-2)^3, (-a)^3.$
12. $x, (-x)^2, (-x)^3.$
13. $x^3, (-x)^3.$
14. $-x^2, (-x)^2.$
15. $2gx, 4gx, -gx^6.$
16. $m^2, -n^2, (-p)^2$

WRITTEN EXERCISES

Multiply:

1. $(-2)^2 \cdot (-3)^3 \cdot (-1)^2.$
2. $(-3ab)^3 \cdot (-3ab)^2 \cdot 5a^2b^3.$
3. $abxy \cdot -21a^3b^5x^my^n.$
4. $31a^rx^s \cdot -15a^{r-1}x^{s-1}.$
5. $-4a^2x^{n-5} \cdot -64a^{n-2}x^5.$
6. $20xy^pz^q \cdot -71x^py^qz^r.$
7. $15a^{m+1} \cdot -12a^{3m}b \cdot 3a^5b^{m-1}.$
8. $17a^mx^{p-2} \cdot -5a^{p-m}x^4.$
9. $-24x^7y^{p-4} \cdot -8x^{r-1}y^5.$
10. $12m^2n \cdot -15mn^8 \cdot -m^pn^s.$
11. $-\frac{1}{8}x^py^qz^s \cdot \frac{1}{7}x^{1-p}y^{1-q}z^{5-s} \cdot -\frac{7}{15}x^5y^6z.$
12. $121s^xt^vy^s \cdot -17s^3t^5z^4 \cdot 0.$

13. Suppose $12a^2b^3c$ to be taken as the product of $4ab^2$ and $3a^2c$. Test this product by letting a, b, and c each equal 1.

14. Also test it by letting $a = 2$, $b = 3$, and $c = 5$. Why did not the test applied in Exercise 13 reveal the error?

115. Signs of Factors. It follows from Sec. 75 that, if the product of two factors is positive, the factors must have like signs; but if the product is negative, the factors must have unlike signs.

For example:

$$8ax = 2a \cdot 4x, \text{ or } (-2a)(-4x).$$
$$-14b^2c = (7b)(-2bc), \text{ or } (-7b)(2bc).$$

WRITTEN EXERCISES

Write a set of two factors for each of the following:

1. $b^2h.$
2. $\frac{1}{2}gt^2.$
3. $\pi r^2h.$
4. $\frac{4}{3}\pi r^3.$
5. $15ax^2y^2.$
6. $-7mn^2p.$
7. $14y^2.$
8. $-48x$

9. mx^2.
10. $-\pi r^2$.
11. $4\pi r^2$.
12. $\frac{1}{2}mv^2$.
13. $\frac{1}{6}\pi d^3$.
14. $6ax^2$.
15. $16ab^2c$.
16. $-8a^3b$.
17. $14br^2$.
18. $21a^3x^3$.
19. $-md^3$.
20. $2\pi rh$.

MULTIPLICATION OF POLYNOMIALS

116. To multiply a polynomial by a monomial *multiply each term of the multiplicand by the monomial and use the signs obtained as the signs of the product.*

For example:

$$\begin{array}{l} 2a-3b+5c \\ 6a \\ \hline 12a^2-18ab+30ac \end{array}$$

117. Distributive Law. The fact that a polynomial is multiplied by multiplying each of its terms separately and taking the algebraic sum of the partial products thus found is called the **Distributive Law of Multiplication.**

The formula, $a(b+c)=ab+ac$, expresses this law in symbols.

ORAL EXERCISES

Read and supply the numbers for the blanks:

1. $\begin{array}{l} 3a+4b \\ 3 \\ \hline (\;)a+(\;)b \end{array}$

2. $\begin{array}{l} 5a^2+2b \\ 10 \\ \hline (\;)a^2+(\;)b \end{array}$

3. $\begin{array}{l} 10x+y \\ 5 \\ \hline (\;)x+(\;)y \end{array}$

4. $\begin{array}{l} 3a+4b \\ 3a \\ \hline (\;)a^2+(\;)ab \end{array}$

5. $\begin{array}{l} 10m+2n \\ 5mn \\ \hline (\;)m^2n+(\;)mn^2 \end{array}$

6. $\begin{array}{l} 4x^2y+xy^2 \\ 3xy \\ \hline (\;)x^3y^2+(\;)x^2y^3 \end{array}$

7. $\begin{array}{l} 6a^2+2b \\ 3b \\ \hline 18(\;)+6(\;) \end{array}$

8. $\begin{array}{l} 5m+3p^2 \\ 3mp \\ \hline (\;)m^2p+(\;)mp^3 \end{array}$

9. $\begin{array}{l} 9x^2+y^2 \\ 5xy \\ \hline 45(\;)+5(\;) \end{array}$

118. To test the work of multiplication, use arbitrary values. The product of the values of the multiplicand and the multiplier must equal the value of the product.

Unity is the easiest number to substitute for the letters; it tests the coefficients and signs in the work of multiplication. It does not, however, test the exponents (see Exercises 13 and 14, p. 77); but this does not impair the test materially, since errors in exponents alone seldom occur.

WRITTEN EXERCISES

Multiply and test:

1. $\begin{array}{l} ax + 4 \\ \underline{3\qquad\quad} \end{array}$

2. $\begin{array}{l} a + b \\ \underline{c\qquad\quad} \end{array}$

3. $\begin{array}{l} 2a + 3b \\ \underline{4c\qquad\quad} \end{array}$

4. $\begin{array}{l} 7ax + 3a \\ \underline{5x\qquad\qquad} \end{array}$

5. $\begin{array}{l} 8a^2 + 2b^2 \\ \underline{ab\qquad\qquad} \end{array}$

6. $\begin{array}{l} 7ax + 3bx^3 \\ \underline{cx\qquad\qquad\quad} \end{array}$

7. $\begin{array}{l} a^2 + 2bx^3 \\ \underline{3b^2x\qquad\quad} \end{array}$

8. $\begin{array}{l} 3m^2 + r^2 \\ \underline{4a\qquad\qquad} \end{array}$

9. $\begin{array}{l} mt^2 + v \\ \underline{vt\qquad\quad} \end{array}$

10. $4x^2(a - 5x)$.
11. $a(2x^3 + 1)$.
12. $a^2b(a^2c - b^2d)$.
13. $2x(3x + 5y)$.
14. $(7x - 5)2a$.
15. $(4x + 5t)2tx$.
16. $6q^2(5q + 18q^3)$.
17. $\frac{1}{2}r(2r^2 + \frac{1}{4}r)$.
18. $6t(t^2 + \frac{1}{2}at)$.

119. Special Tests. Certain polynomials have properties which aid in testing the work of multiplication.

Thus, in $(x + y)(x^2 - xy + y^2) = x^3 + y^3$, each term of the first factor is of the first degree, and each term of the second is of the second degree; hence, if each term of the product were not of the third degree, the product would be incorrect.

120. Expressions all of whose terms are of the same degree are called **homogeneous**; when factors are homogeneous, their product is homogeneous, and its degree is the sum of the degrees of its factors.

ORAL EXERCISES

1. Without multiplying, state the degree of each term in the product of $a + b$ and $a^2 + b^2$.

2. Similarly for $x^2 + xy$ and $x + y$. Also for $mn + n^2$ and $m^2 + n^2$.

3. Without multiplying, determine whether or not $x^3 + xy + y^3$ is the product of $x^2 + xy + y^2$ and $x + y$.

WRITTEN EXERCISES

Multiply and test first by seeing whether the product is homogeneous. If it is homogeneous, test further by substituting arbitrary values:

1. $(a^3 + 2b^3)(a^3 - 3b^3)$.
2. $(x + y)(x^2 - 3xy + 5y^2)$.

3. $(x^3 - y^2)(x^2 + 6xy - y^2)$.
4. $(a + x)^2(a^2 - x^2)$.
5. $(a^2 - 5ax + x^2)(x^3 + 2a^3)$
6. $(x^2 - 2xy + y^2)(x^2 + y^2)$.

121. Removal of Parentheses. *If a parenthesis used to indicate multiplication is removed, the multiplication must be performed.*

Thus: $7a - 5(9a - 4b) = 7a - 45a + 20b = -38a + 20b$.

WRITTEN EXERCISES

Remove parentheses and unite terms as much as possible:

1. $3 + 5(6 - 4)$.
2. $5x + 3(11x - 5)$.
3. $7(4a - 2b) + 10b$.
4. $6y - 7(4y + 3t)$.
5. $11 - 3(7 - 2x)$.
6. $4a - 12(7 - 6a)$.
7. $9(a - x) - a(5 + x)$.
8. $-5(2x - 1) + 3(4x - 8)$.
9. $a(b + c) - c(a + b) + b(a - c)$.
10. $2m - \{4m + 7(6m - 1)\}$.
11. $p\{4r - 3r(1 - a) + 5ar\}$.
12. $x^2\{x^2 - 2a(2x - 3a)\} - a^3(4x - a)$.
13. $-10\{x - 6[x - (y - z)]\} + 60\{y - (z + x)\}$.

122. The multiplication of literal numbers is similar to the multiplication of numbers expressed by figures.

For example:

MULTIPLICATION WITH FIGURES	MULTIPLICATION WITH LETTERS	TEST
32	$3a + 2b$	5
$\underline{14}$	$\underline{a + 4b}$	$\underline{5}$
$128 = 4 \times 32$	$3a^2 + 2ab \qquad = a(3a + 2b)$	
$\underline{320} = 10 \times 32$	$\underline{12ab + 8b^2} = 4b(3a + 2b)$	
$448 = 14 \times 32$	$3a^2 + 14ab + 8b^2 = (a + 4b)(3a + 2b)$	25

123. To multiply by a polynomial, *multiply by each term of the polynomial, add like terms, and use the signs obtained as the signs of the result.*

Thus:

$$\begin{array}{r} 2a^2 + 5a - 2 \\ \underline{a^2 - 3a + 1} \\ 2a^4 + 5a^3 - 2a^2 \\ -6a^3 - 15a^2 + 6a \\ \underline{2a^2 + 5a - 2} \\ 2a^4 - a^3 - 15a^2 + 11a - 2 \end{array}$$

WRITTEN EXERCISES

Multiply and test:

1. $\begin{array}{l} a+b \\ \underline{a+b} \end{array}$
2. $\begin{array}{l} a+b \\ \underline{a-b} \end{array}$
3. $\begin{array}{l} a-b \\ \underline{a-b} \end{array}$
4. $\begin{array}{l} c+1 \\ \underline{c-1} \end{array}$
5. $\begin{array}{l} x+2 \\ \underline{x+2} \end{array}$
6. $\begin{array}{l} z^2+5 \\ \underline{z^2+5} \end{array}$
7. $\begin{array}{l} 2a-1 \\ \underline{2a-1} \end{array}$
8. $\begin{array}{l} 12+x \\ \underline{12-x} \end{array}$
9. $\begin{array}{l} 3y-5 \\ \underline{2y+4} \end{array}$
10. $\begin{array}{l} x+a \\ \underline{x+b} \end{array}$
11. $\begin{array}{r} 3a+x \\ \underline{a+b} \end{array}$
12. $\begin{array}{r} 4a+5 \\ \underline{x-a} \end{array}$
13. $\begin{array}{r} m+3 \\ \underline{3m+2} \end{array}$
14. $\begin{array}{l} m^2-n^2 \\ \underline{m^2+n^3} \end{array}$
15. $\begin{array}{l} x^2+1 \\ \underline{x^2-1} \end{array}$
16. $\begin{array}{l} 2a+b \\ \underline{a+2b} \end{array}$
17. $\begin{array}{r} 2a-\quad b \\ \underline{c-3a} \end{array}$
18. $\begin{array}{l} 3x+2y \\ \underline{2x+3y} \end{array}$
19. $\begin{array}{l} 3ab+4b^2 \\ \underline{2ab-3b^2} \end{array}$
20. $\begin{array}{l} x^2+3x-1 \\ \underline{x\ +3\qquad} \end{array}$
21. $\begin{array}{l} x^2-4x+3 \\ \underline{x\ -2\qquad} \end{array}$
22. $\begin{array}{l} x^2-ax+b \\ \underline{x\ -c\qquad} \end{array}$
23. $\begin{array}{l} x^2-ax+b \\ \underline{3x+a\qquad} \end{array}$
24. $\begin{array}{l} t^2+tu+u^2 \\ \underline{t\ -u\qquad} \end{array}$
25. $\begin{array}{l} a+b-c \\ \underline{a-b+c} \end{array}$
26. $\begin{array}{l} x^2+y^2-z^2 \\ \underline{x\ +y-z} \end{array}$
27. $\begin{array}{r} xy+yz+xz \\ \underline{x-y\ +z} \end{array}$

124. *To find the product of expressions involving literal coefficients and exponents, find the product of the coefficients and add the exponents as in numerical cases.*

EXAMPLES

1. Multiply ax and $(a-b)x$.

$$\begin{array}{r} (a-b)x \\ ax \\ \hline a(a-b)x^2 \end{array}$$

2. Multiply x^n+1 by x^n-2.

$$\begin{array}{l} x^n+1 \\ x^n-2 \\ \hline x^{2n}-x^n-2 \end{array}$$

WRITTEN EXERCISES

Multiply:

1. $(5a + x)(5a - x)$.
2. $(x - y)(x^m - y^m)$.
3. $(x^m + y^n)(x + y)$.
4. $(4x^m + 3y^n)(4x^m - 3y^n)$.
5. $(x^m - y^n)(x^{2m} - y^{2n})$.
6. $(a^{2n} + c^{2n})(a^{mn} + c^{mn})$.
7. $z^n(z^r + z^s)$.
8. $(xc - 1)cx$.
9. $(a - b)y \cdot aby$.
10. $(c + d)x \cdot (c - d)x$.
11. $(a^m + c)(a^m - c)$.
12. $(a - 3ab^3)(a + 3ab^3)$.
13. $(a + b)x^n \cdot (a - 2b)xy^p$.
14. $(m - n - 2p)x^r \cdot -3m^2npx^s$.
15. $(m^2 - n^2)x^4y^5 \cdot (m^2 + n^2)x^2y^2$.
16. $(p^2 - p + 1)x^{2a}y^{2b} \cdot (p + 1)x^by^a$.

REVIEW

ORAL EXERCISES

State the products:

1. $\begin{array}{r} 4x \\ \underline{9x} \end{array}$

2. $\begin{array}{r} 8y \\ \underline{-3y} \end{array}$

3. $\begin{array}{r} 7a \\ \underline{4a} \end{array}$

4. $\begin{array}{r} -6t \\ \underline{-2t} \end{array}$

5. $\begin{array}{r} -9m \\ \underline{3m} \end{array}$

6. $\begin{array}{r} 5ab \\ \underline{7ac} \end{array}$

7. $\begin{array}{r} -2ax \\ \underline{+5x^2} \end{array}$

8. $\begin{array}{r} -6ay \\ \underline{-4ay} \end{array}$

9. $\begin{array}{r} 7x^2y \\ \underline{-\quad xy} \end{array}$

10. $\begin{array}{r} -9am \\ \underline{\tfrac{1}{3}ar} \end{array}$

WRITTEN EXERCISES

Multiply and test:

1. $\begin{array}{l} ax + 3 \\ \underline{ax + 5} \end{array}$

2. $\begin{array}{l} 4ab + c \\ \underline{2ab + 3c} \end{array}$

3. $\begin{array}{l} y + x + b \\ \underline{y + 5x - b} \end{array}$

4. $\begin{array}{l} a - b - c \\ \underline{a - b - c} \end{array}$

5. $\begin{array}{l} 1 - x + x^2 \\ \underline{1 + x - x^2} \end{array}$

6. $\begin{array}{l} a + 3x - 1 \\ \underline{x + 2a + 1} \end{array}$

7. $\begin{array}{l} x^3 - 7x^2 + 5x - 3 \\ \underline{2x - 4 \qquad\qquad\quad} \end{array}$

8. $\begin{array}{l} a^3 - 3a^2y + 3ay^2 - y^3 \\ \underline{a - y \qquad\qquad\qquad\quad} \end{array}$

9. $\begin{array}{l} x^2 + 3px - 4p^2 \\ \underline{2x^2 - 7px - p^2} \end{array}$

10. $\begin{array}{l} x^3 + 3x^2y + 3xy^2 + y^3 \\ \underline{x + y \qquad\qquad\qquad\quad} \end{array}$

11. $\begin{array}{r} x-5 \\ \underline{x+6} \end{array}$

12. $\begin{array}{r} 2x+3 \\ \underline{x-1} \end{array}$

13. $\begin{array}{r} 3x+5 \\ \underline{2x-4} \end{array}$

14. $\begin{array}{r} 6t-3u \\ \underline{t+2u} \end{array}$

15. $\begin{array}{r} 3x^2+5 \\ \underline{x-9} \end{array}$

16. $\begin{array}{l} x^2+x+1 \\ \underline{x\ \ +1\qquad} \end{array}$

17. $\begin{array}{r} x^2-2x+1 \\ \underline{x-1} \end{array}$

18. $\begin{array}{r} x^2+3x+2 \\ \underline{x^2+\ \ x+2} \end{array}$

19. $\begin{array}{l} 2x^2 -- x+1 \\ \underline{2x\ \ -5\qquad} \end{array}$

20. $(2a^n+3b^n)(2a^n+3b^n)$.

21. $(r^{p-1}-s^{p-1})(r-s)$.

22. $(4z^{3n}-2^{2n}+z^n-1)(3z^n+1)$.

23. $(3x^m-y^r)(3x^m+y^r)$.

24. $(x-1)(x-2)(x-3)$.

25. $(x+6)(x-5)(x-3)$.

26. $(a^3-a^2-1)(a+1)$.

27. $(m^2-m+1)(m+1)$.

28. $(x^4-x^2+1)(x^2+6)$.

29. $x^2y^s(x^ny^r-x^9y^4)$.

30. $(a-b)x\cdot(a+b)x$.

31. $ab\cdot(c-1)b$.

32. $ax^2\cdot(a-b)x^2\cdot(b+c)x^2$.

33. $(x^2y-xy^2)(x+y)$.

34. $(m^5+m^3+1)(m^2+1)$.

35. $(y^4+y+2)(y^3-1)$.

36. $(y^3-3y+5)(y^2+10)$.

37. $(rs-r^2s^2)(rs+r^2s^2)$.

SUMMARY

The following questions summarize the definitions and processes in this chapter:

1. State how to find the product of two monomials. Sec. 110.

2. State the *Law of Exponents in Multiplication.* Sec. 112.

3. State how to find the product of several monomials. Sec. 113.

4. State how to multiply a polynomial by a monomial Sec. 116

5. State the *Distributive Law of Multiplication.* Illustrate this law in symbols. Sec. 117.

6. State how to test the work in multiplication. Secs. 118, 119.

7. State how to multiply one polynomial by another. Sec. 123.

CHAPTER IX

DIVISION

DIVISION OF MONOMIALS

125. Preparatory.

1 4 lb. ÷ 2 = ( ) lb. 4 yd. ÷ 2 = ( ) yd. $4y \div 2 = (\ )y$.

2. 6 oz. ÷ 3 = ( ) oz. 6 ft. ÷ 3 = ( ) ft. $6f \div 2 = (\ )f$.

3. $3a \cdot (\ ) = 6a^2b$, then $6a^2b \div 3a = ?$

4. $8b^2 \cdot (\ ) = 16ab^3$, then $16ab^3 \div 8b^2 = ?$

5. $a^2b \cdot (\ ) = a^2bcd$, then $a^2bcd \div a^2b = ?$

126. Division. Division is the process of finding one of two factors when the product and the other factor are given. Division is thus the inverse of multiplication.

The problem $18a^3b^7c \div 3a^2b^4c$ means to find the number by which $3a^2b^4c$ must be multiplied to produce $18a^3b^7c$.

To do this:

3 must be multiplied by 6 to produce 18.

a^2 must be multiplied by a to produce a^3.

b^4 must be multiplied by b^3 to produce b^7.

c multiplied by 1 produces c.

Hence, the quotient of $18a^3b^7c \div 3a^2b^4c$ is $6 \cdot a \cdot b^3 \cdot 1$, or $6ab^3$.

Test. The product of the divisor and the quotient must equal the dividend.

127. Zero cannot be used as a divisor.

This may be seen by reference to Sec. 114.

For, $0 \cdot 3 = 0$ and $0 \cdot 5 = 0$.

Therefore, $0 \cdot 3 = 0 \cdot 5$.

Now if we could divide both numbers by 0, the result would be $3 = 5$, which is not true.

WRITTEN EXERCISES

Divide and test:

1. $65\,a^2b \div 13\,ab$.
2. $48\,x^2y \div 12\,xy$.
3. $63\,m^2n^2 \div 21\,mn$.
4. $96\,a^3b^2 \div 12\,ab$.
5. $45\,p^2q^3 \div 15\,p^2q^2$.
6. $120\,abc \div 20\,a$.
7. $\frac{1}{2}\,mv^2 \div mv$.
8. $\frac{1}{2}\,gt^2 \div \frac{1}{2}\,t$.
9. $\frac{4}{3}\,\pi r^3 \div \pi r^2$.
10. $\frac{1}{6}\,\pi d^3 \div \frac{1}{6}\,\pi d$.

11. Find the numbers to fill the blanks:

	(1)	(2)	(3)	(4)	(5)
Dividend:	$12\,x^2$	$27\,ab$	$33\,ay$	$42\,p$	$36\,mn$
Divisor:	$3\,x$	——	——	$7\,p$	$9\,m$
Quotient:	——	$3\,b$	$11\,a$	——	——

128. The equation:

$$\text{Dividend} = \text{Divisor} \times \text{Quotient}$$

remains true if the two numbers, dividend and divisor, are multiplied (or divided) by the same number.

In other words:

The quotient is not altered, if both dividend and divisor are multiplied or divided by the same number.

Canceling may be used as in arithmetic:

Thus, $$\frac{8\,ab^6}{4\,ab^2} = \overset{2\,b^4}{\frac{\cancel{8}\,\cancel{a}\cancel{b^6}}{\cancel{4}\,\cancel{a}\cancel{b^2}}} = 2\,b^4.$$

Exponents must not be canceled. E.g. if we should cancel the exponent 2 from the exponent 6, the result would be b^3 instead of b^4.

Taking out factors results in subtracting not dividing exponents.

129. Law of Exponents. Since $a^5 = a^3 \cdot a^2$, it follows by dividing both members by a^2 that $\frac{a^5}{a^2} = a^3$.

Likewise, from $a^{m+r} = a^m \cdot a^r$, it follows by dividing both members by a^r that $\frac{a^{m+r}}{a^r} = a^m$.

Accordingly, the exponent of any letter in the quotient is the difference between the exponent of that letter in the dividend and its exponent in the divisor.

For example:

$$\frac{a^{12}}{a^7} = a^5, \text{ also } \frac{a^{3n}}{a^n} = a^{2n},$$

$$\frac{a^{n+2}}{a^2} = a^n, \text{ also } \frac{(ab)^5}{(ab)^2} = (ab)^3.$$

ORAL EXERCISES

Divide:

1. $a^5 \div a^3$.
2. $a^7 \div a^5$.
3. $b^{10} \div b^3$.
4. $b^7 \div b^4$.
5. $a^3b \div a^2$.
6. $a^4b^2 \div a^2b$.
7. $m^7 \div m^2$.
8. $\frac{14\,a^3b}{2\,a}$.
9. $\frac{12\,a^{4x}}{10\,a^{3x}}$.
10. $\frac{a^{4m}}{a^{2m}}$.
11. $\frac{3\,a^2b^q}{3\,ab^q}$.
12. $\frac{b^{5n+3}}{b^3}$.
13. $\frac{25\,m^{2x}n}{5\,m^x}$.
14. $\frac{x^3y^{3n}}{xy^n}$.
15. $\frac{p^2qr}{pqr}$.
16. $\frac{65\,x^3y^5}{13\,xy^3}$.
17. $\frac{18\,abx^3y^{2p}}{9\,abxy^p}$.
18. $\frac{6\,a^2b^{x+2}}{3\,ab^2}$.
19. $\frac{20\,m^2n^2q}{6\,mn^2q}$.

130. To find the quotient of two monomials:

1. *Find the sign of the quotient,* by the rule of signs. (Sec. 77.)

2. *Find the numerical coefficient,* by dividing the given numerical coefficients.

3. *Find the literal part,* by dividing the given literal parts.

Thus, in $18\,a^3b \div -3\,ab$, the sign is $-$, the numerical coefficient is $18 \div 3$ or 6, the literal part is $a^3b \div ab$ or a^2; $\therefore$ the quotient is $-6\,a^2$.

WRITTEN EXERCISES

Divide:

1. $\frac{2\,a^2cx^2y}{-2\,axy}$.
2. $\frac{-\,abcx^2y}{-\,cx}$.
3. $\frac{-\,5\,abc^2}{-\,5\,bc}$.
4. $\frac{10\,a^3x^2y^3}{-\,5\,ax^2}$.
5. $\frac{10\,m^3n^3}{-\,\frac{1}{6}\,mn^2}$.
6. $\frac{1.4\,a^5b}{.7a^3b}$.
7. $\frac{a^{7m}b^{4c}}{a^{3m}b^c}$.
8. $\frac{18\,a^3b^3}{3\,a^2b}$.
9. $\frac{24\,abc}{-\,3\,b}$.

10. $\dfrac{1.5\,x^3yz}{-5\,x^2y}$.

11. $\dfrac{m^2n^2p^2}{-n\cdot -p}$.

12. $\dfrac{-a^4b^3c^3}{-a^2\cdot -bc}$.

13. $\dfrac{-35\,b^6c^6}{7\,b^3c^3}$.

14. $\dfrac{10^{6n}x^{3n+8}}{10^{2n}x^{3n+4}}$.

15. $\dfrac{-2\,a^3x^3}{\frac{1}{2}\,ax^2}$.

16. $\dfrac{-2\,p^3q^4r^4t^4}{-\frac{1}{5}\,pqrt^3}$.

17. $\dfrac{a^4b^2c^2x}{-a^2bcx}$.

18. $\dfrac{-16\,a^2x^2y^3}{4\,y}$.

19. $\dfrac{27\,xy^2z^4}{-3\,z^3}$.

20. $\dfrac{16\,b^6cx^6}{-4\,b^3cx^3}$.

21. $\dfrac{54\,m^2p^3x^5}{-9\,mp^2x^3}$.

22. $\dfrac{a^{n+7}b^{3n+6}c^{2n+9}}{-a^4b^3c^7}$.

23. $3\,a^4b^4c^3 \div -abc$.

24. $2\cdot 5\,a^2x^3y \div -5\,ax^2y$.

25. $55\,a^2x^2 \div -11\,a^2x$.

26. $\dfrac{-18\,x^{a+5}y^{2a+7}}{3\,x^4y^{a+3}}$.

27. $\frac{4}{3}\,\pi r^3 \div \frac{1}{3}\,\pi r$.

28. $\frac{1}{6}\,\pi d^3 \div \frac{1}{2}\,\pi d^2$.

29. $-27\,x^2y^2z^2 \div 3\,xyz^2$.

DIVISION OF POLYNOMIALS

131. PREPARATORY.

1. $2\overline{)\,4 \text{ bu. } 2 \text{ pk.}}$ $\quad$ $2\overline{)\,4 \text{ lb. } 2 \text{ oz.}}$ $\quad$ $2\overline{)\,4\,l + 2\,z}$

   ( ) bu. ( ) pk. $\quad$ ( ) lb. ( ) oz. $\quad$ ( )l + ( )z

2. $3\overline{)\,9 \text{ ft. } 6 \text{ in.}}$ $\quad$ $3\overline{)\,9 \text{ mi. } 6 \text{ rd.}}$ $\quad$ $3\overline{)\,9\,m + 6\,r}$

   ( ) ft. ( ) in. $\quad$ ( ) mi. ( ) rd. $\quad$ ( )m + ( )r

MULTIPLICATION	DIVISION
Since (1)	then, (2)
$$\begin{array}{r} a - bd + 5\,c \\ m \\ \hline ma - mbd + 5\,mc \end{array}$$	$$\begin{array}{l} m\overline{)\,ma - mbd + 5\,mc} \\ \quad a - bd + 5\,c \end{array}$$

132. *Accordingly, to divide a polynomial by a monomial, divide each term of the polynomial by the monomial, and use the signs obtained as the signs of the quotient.*

WRITTEN EXERCISES

Divide and test:

1. $(6\,a^2 + 3\,a) \div 3$.
2. $(12\,ab + 4\,b) \div 4$.
3. $(6\,a^2 + 3\,a) \div a$.
4. $(12\,ab + 5\,b) \div b$.
5. $(6\,a^2 + 3\,a) \div 3\,a$.
6. $(12\,ab + 4\,b) \div 4\,b$.

7. $5a^2 - 4ab + 4a$ by a.　　9. $(x^2y + xy^2) \div xy$.

8. $a^6 - 5a^5 + 3a^4$ by a^2.　　10. $(3x^2y - 6xy) \div 3xy$.

11. $25a^2 + 10ab + 5b^2$ by 5.

12. $27a^2b - 9ab^2 + 9a^2b^2$ by $9ab$.

13. $(12mn + 27mn^2p) \div 3mn$.

14. $(6x^2y - 4x^2z + 6xyz) \div 2x$.

15. $4x^4y^4 - 8x^3y^3 + 6xy^3$ by $-2xy$.

16. $-3a^2 + \frac{9}{2}ab - 3ac$ by $-\frac{3}{2}a$.

17. $x^3y - 3x^2y + 9xy^2$ by $3xy$.

18. $(x^3y - 3x^2y^3 + 9xy^2) \div 3xy$.

19. $(a^2c^2 - 2abc^2 + 3ac^3) \div ac^2$.

20. $a^2c^2 - 2abc^2 + 3ac^3$ by $-ac^2$.

21. $(5a^3b^3 - 35a^2b^2c^2 + 2ab^2c^4) \div 5ab$.

22. $(2m^2n^2 - 3mn^3 + 4m^2n - n^4) \div 3n$.

23. $(a^3x^3y - 3a^2bx^2y + 3ab^2xy^2 - a^2b^3xy^3) \div axy$.

24. $x^{7n} + 4x^{9n}y^2 + 3x^{8n}y^{2n}$ by x^{3n}.

25. $a^{7c+5} + a^{4c+8} + a^{11c+12}$ by a^4.

26. $10^{4n+12} + 10^{8n+9} + 10^{6n+6}$ by 10^{4n+3}.

133. To Divide by a Polynomial. This process is seen best from examples.

1. Divide $x^2 + 3xy + 2y^2$ by $y + x$.

$$\begin{array}{rl}
 & \text{QUOTIENT} \\
 & \underline{x + 2y} \\
\text{DIVISOR } x + y) & x^2 + 3xy + 2y^2 \\
 & \underline{x^2 + xy} \\
 & \qquad 2xy + 2y^2 \\
 & \qquad \underline{2xy + 2y^2}
\end{array}$$

Arrange the terms of the divisor and the dividend according to the powers of the same letter (x in this example).

Divide the first term of the dividend by the first term of the divisor.

The result (in this example, x) is the first term of the quotient.

Multiply the entire divisor by this term and subtract.

Divide the first term of the remainder by the first term of the divisor. The result (in this example, $2y$) is the second term of the quotient.

Multiply the entire divisor by this term and subtract.

Continue the process until a remainder zero is reached.

Cases in which such a remainder cannot be reached are treated later.

Test. Multiply the quotient by the divisor. If the work has been correctly done (including the multiplication), the result will equal the dividend. Or, substitute arbitrary values and proceed as in previous cases.

2. Divide $6a^3-17a^2b+16b^3$ by $3a-4b$.

SOLUTION

$$\begin{array}{rllll}
 & 2a^2 & -3ab & -4b^2 & \\ \cline{2-5}
3a-4b) & 6a^3 & -17a^2b & & +16b^3 \\
 & 6a^3 & -8a^2b & & \\ \cline{2-3}
 & & -9a^2b & & \\
 & & -9a^2b & +12ab^2 & \\ \cline{3-5}
 & & & -12ab^2 & +16b^3 \\
 & & & -12ab^2 & +16b^3 \\ \cline{4-5}
\end{array}$$

TEST (Substitute $a=b=1$)

Divisor × Quotient = Dividend

$$(3-4)(2-3-4)=6-17+16$$
$$-1\cdot(-5)=5$$
$$5=5$$

WRITTEN EXERCISES

Divide and test:

1. $a^2+2ab+b^2$ by $a+b$.
2. a^2+3a+2 by $a+1$.
3. $x^2-11x+30$ by $x-5$.
4. $x^2+4xy+4y^2$ by $x+2y$.
5. $3c^2+7cd+2d^2$ by $d+3c$.
6. $6a^2-7a-3$ by $2a-3$.
7. $2x^2-xy-3y^2$ by $x+y$.
8. $3a^2+ab-2b^2$ by $3a-2b$.
9. $6m^2+mn-2n^2$ by $3m+2n$.
10. $12y^2+19y-21$ by $3y+7$.
11. $a^3+3a^2b+3ab^2+b^3$ by $a+b$.
12. $96a^2-4ab-15b^2$ by $12a-5b$.
13. $a^3+3a^2b+3ab^2+b^3$ by $a^2+2ab+b^2$.
14. $a^4+a^2b^2+b^4$ by a^2-ab+b^2.
15. x^3-2x^2-2x+1 by $x+1$.
16. $3p^2-115-8p$ by $p+5$.
17. $3a^4-2a-1865$ by $a-5$.
18. $14y^4-13y-43y^2+32y^3+3$ by $7y^2-5y-3$.
19. $21a^3-4a^2-16-46a$ by $4a+2-3a^2$.
20. $\dfrac{1+a^6}{1-a^2+a^4}$.
21. $\dfrac{a^8-b^8}{a^2+b^2}$.

22. $\dfrac{x^{2p}+8x^{p}+16}{x^{p}+4}$.

23. $\dfrac{9m^2+12mn+4n^2}{3m+2n}$.

24. $\dfrac{a^4+4b^4}{a^2+2ab+2b^2}$.

25. $\dfrac{6a^2b^2-ab^3-12b^4}{3ab+4b^2}$.

26. $\dfrac{4x^4y^4+1}{2x^2y^2-2xy+1}$.

27. $\dfrac{a^3+b^3}{a^2-ab+b^2}$.

28. $\dfrac{a^3-b^3}{a^2+ab+b^2}$.

29. $\dfrac{x^6-4x^4y^2+4x^2y^4-y^6}{x^2-y^2}$.

30. $\dfrac{a^5-a^4-11a^3+16a^2-2a-3}{a^2-4a+3}$.

31. $\dfrac{6x^4+17x^3-19x-4}{2x^3+3x^2-4x-1}$.

32. $\dfrac{x^{3n}+3x^{n}y^{2n}-x^{n}y^{n}-3y^{3n}}{x^{n}-y^{n}}$.

134. Remainders. In algebra, as in arithmetic, the division may not be exact. That is, no remainder may be zero, however far the division is carried.

$$\begin{array}{r|l}
 & x^3-x^2+x-1 \\ \hline
x+1 & x^4+1 \\
 & \underline{x^4+x^3} \\
 & -x^3+1 \\
 & \underline{-x^3-x^2} \\
 & x^2+1 \\
 & \underline{x^2+x} \\
 & -x+1 \\
 & \underline{-x-1} \\
 & 2
\end{array}$$

Thus, in the example at the right, there is a remainder, 2. The division might be continued, the next term of the quotient being $\frac{2}{x}$; but it is customary to stop as soon as a remainder is reached that is of lower degree than the divisor. The integral part of the quotient has now been found; it is called the integral quotient.

By using the fractional form to indicate the division of the remainder, the result of the above division may be expressed thus:

$$\frac{x^4+1}{x+1}=x^3-x^2+x-1+\frac{2}{x+1}.$$

The right member of the equation is called the complete quotient.

Test. Dividend = divisor × integral quotient + the remainder.

Substituting $x = 1$.

Dividend = Divisor × Integral Quotient + Remainder.

$$1+1=(1+1)(1-1+1-1)+2.$$
$$2=2\cdot 0+2.$$

WRITTEN EXERCISES

Find the quotient and remainder:

1. $(x^3-1)\div(x+1)$.
2. $(a^2+x^2)\div(a+x)$.
3. $\dfrac{x^5+x+1}{x^2-1}$.
4. $\dfrac{x^4-2x^2+1}{x^2+1}$.
5. $(3x^2-5x+2)\div(x-4)$.
6. $(8y^2+7y-1)\div(2y+3)$.
7. $\dfrac{6ax-9ay-4bx+8by}{3a-2b}$.
8. $\dfrac{2a^3-2a^2-6a+4}{2a-3}$.

REVIEW

ORAL EXERCISES

Divide and test:

1. $\dfrac{-32ax^4}{4x^3}$.
2. $\dfrac{-42m^6y^3}{6m^2y}$.
3. $\dfrac{144a^5x^4}{-16ax^3}$.
4. $\dfrac{-14641p^2q}{-11p}$.
5. $\dfrac{t^2-14t+49}{t-7}$.
6. $\dfrac{1-16y+64y^2}{1-8y}$.
7. $\dfrac{ax^3+bx^2}{x^2}$.
8. $\dfrac{x^2-y^2}{x-y}$.
9. $\dfrac{x^2-y^2}{x+y}$.
10. $\dfrac{a^2-x^2}{a+x}$.
11. $\dfrac{m^2p^2-n^2q^2}{mp+nq}$.
12. $\dfrac{x^4-y^4}{x^2-y^2}$.
13. $\dfrac{m^4-n^4}{m^2+n^2}$.
14. $\dfrac{x^2y^2-m^2n^2}{xy-mn}$.
15. $\dfrac{x^{2m}-y^{2r}}{x^m-y^r}$.
16. $(14ax^2+6ax)\div 2a$.
17. $(25t^3-40t)\div -5$.
18. $(18x^2z^3-24x^3z^2)\div -6xz^2$.
19. $(a^2bc-ab^2c+2abc^2)\div abc$.
20. $(-27m^{2x}y^{2a}-9m^{5x}y^{3a})\div 9m^xy^a$.
21. $(-25p^{2a}q^{3r}+10p^{3a}q^{2r})\div 5p^aq^r$.

WRITTEN EXERCISES

Find the quotient and remainder, if any:

1. p^4-16 by $p+2$.
2. a^4-10a^2+25 by a^2-5.
3. $6y^2+2y-20$ by $2y+4$.

4. $2a^2 + 5ab + 2b^2$ by $a + 2b$.

5. $6a^2b^2 - ab^3 - 12b^4$ by $2ab - 3b^2$.

6. $x^2 + ax + bx + ab$ by $x + a$.

7. $m^4 - m^2p^2 + m^2p^3 - p^5$ by $m^2 - p^2$.

8. $x^3 + 6x^2 + 8x - 3$ by $x^2 + 3x - 1$.

9. $2ac - bc - 6a^2 + 3ab$ by $c - 3a$.

10. $1 - x^2 + 2x^3 - x^4$ by $x^2 - x + 1$.

11. $\dfrac{x^5 - 1}{x^2 - 1}$.

12. $\dfrac{x^3 - 7x^2 + 3x - 1}{x - 3}$.

13. $\dfrac{a^4 - 5a^2 + 4}{a + 2}$.

14. $\dfrac{x^3 - 2x^2 + x - 4}{x - 3}$.

15. $\dfrac{x^4 - 2x^2 + 3x}{x^2 - 1}$.

16. $\dfrac{8 - 12a + 6a^2 - a^3}{2 - a}$.

17. $\dfrac{15a^2b + 6ab^2 + 8a^3 + 3b^3}{8a - b}$.

18. $\dfrac{-7r^2 + 3r^3 + 5 + r^4}{r^2 - 5r + 2}$.

19. $\dfrac{18x^4 - 24x^3 + 38x^2 - 68x + 32}{3x - 2}$.

Divide:

20. $z^4 - 1$ by $z - 1$.

21. $x^6 - x^3y^3$ by $x - y$.

22. $a^4 - 2a^2 + 1$ by $a - 1$.

23. $a^2 + 2ab + b^2 - c^2$ by $a + b + c$.

24. $m + \dfrac{1}{p}$ by $\dfrac{1}{m} + p$ to four terms.

25. $x^4 + x^2y^2 + y^4$ by $x^2 + y^2 - xy$.

26. $\dfrac{1}{a^2} + 1 + a^2$ by $\dfrac{1}{a^2} - \dfrac{1}{a} + 1$.

27. $x^2 - y^2 - 2y - 1$ by $x + y + 1$.

28. $6x^2 - 2xy - 3x + y$ by $3x - y$.

29. $x^{3m} + y^{3n}$ by $x^m + y^n$.

30. $6r^6 - 18r^5 + 18r^4 - 3r^3 - 9r^2 + 9r - 3$ by $2r^3 + 1$.

31. $a^{4x} + 4a^{3x}b + 6a^{2x}b^2 + 4a^xb^3 + b^4$ by $a^{2x} + 2a^xb + b^2$.

SUMMARY

These questions summarize the definitions and processes in this chapter:

1. State in the form of an equation the relation between the dividend, divisor, and quotient. Sec. 128.
2. State the *Law of Exponents* in division. Sec. 129.
3. State how to find the quotient of two monomials. Sec. 130.
4. Explain how to divide a polynomial by a monomial. Sec. 132.
5. State how to divide by a polynomial. Sec. 133.
6. State how to test the work of division. Secs. 126, 133.
7. When is the remainder reached in the process of division? Sec. 134.

HISTORICAL NOTE

After solving many exercises in the four processes with polynomials, algebra may seem to consist mostly of such manipulations; but we shall see as we proceed that the solution of equations becomes more and more prominent and that the facility we have gained in the use of symbols is a help in handling equations. In fact, the word "algebra" itself is connected in its origin with the equation. There lived in Bagdad, Arabia, about 825 A.D., a famous mathematician known from his birthplace, Kharezm, as Al-khowarazmi, and in the title of his work signifying "The science of transposing and combining in solving equations," there appeared the Arabic word *Al-Jabr* to denote these processes. When this manuscript was translated into Latin in the thirteenth century, the word *Al-Jabr* was transferred as *algebræ*, from which we have our word "algebra." Thus, *algebra*, instead of meaning originally "science of symbols," as we might imagine, means "the science of processes in equations."

Although Diophantos (300 A.D.) could find the product of two binomials, the knowledge of how to multiply or divide one polynomial by another is comparatively recent. Ordinary algebraic division was developed in the seventeenth century, and Newton, in his *Arithmetica Universalis* (1707), showed how arranging the terms in both dividend and divisor according to the descending powers of the same letter facilitated the work.

CHAPTER X

EQUATIONS

135. The parenthesis is often used in equations to indicate multiplication.

EXAMPLES

1. Solve: $(x+1)(x-5)=x(x-1)$. *(1)*

Removing the parentheses, $x^2-4x-5=x^2-x$. *(2)*

Then, $-3x=5$;

and $x=-\frac{5}{3}$. *(3)*

Test. $(-\frac{5}{3}+1)(-\frac{5}{3}-5)=\frac{40}{9}=-\frac{5}{3}(-\frac{5}{3}-1)$.

2. Solve: $(ax-1)x+2-b=ax^2-c$. *(1)*

Removing the parenthesis, $ax^2-x+2-b=ax^2-c$. *(2)*

Uniting terms, $-x+2-b=-c$. *(3)*

Then, $-x=-c+b-2$,

and $x=c-b+2$. *(4)*

Test by substitution.

WRITTEN EXERCISES

Solve:

1. $x(x-3)+1-x(x-5)=0$.
2. $x^2+3-x(x+4)=15$.
3. $x^2(x-1)-x^3+x^2-2x=12$.
4. $5(x-4)=6(x+1)$.
5. $-3(x+7)=2(1-3x)$.
6. $a(x-b)=3ab$.
7. $(x-4)(x+4)=x^2-8x$.
8. $(x-1)^2=(x-3)^2$.
9. $(x-5)(x+8)=(x-7)^2$.
10. $y(9y-5)=(3y+1)^2$.
11. $t^2-1=(t+4)^2$.
12. $15p=-29-(4-4p)$.
13. $1-(16+7w)=8w$.
14. $2x-(7x-18)=4x$.
15. $(x+5)(2x-1)=x(2x+4)$.
16. $(2x+1)(2x-1)=x(4x-2)$.
17. $3x(6x+5)=18x^2-(x+32)$.

18. $4y(y-1)=(2y-1)(2y+1)$.

19. $4x+8=2-(9x+20)$.

20. $4s=7-(s-3)$.

21. $(x+2)(x-5)=x(x-1)$.

22. $(cx+2)x=cx^2-a$.

23. $ax(x-5)=ax^2-1$.

24. $(a+b)x=cx+5$.

25. $(x-1)(ax+b)=ax^2+b$.

26. $(m+1)x-px=q$.

27. $p(x-5)+4p=1$.

28. $(x-1)(x+2)=(x-p)(x+q)$.

29. $(ax+b)(c+d)=f$.

30. $(ax-1)(bx-1)=(ax+b)(bx-c)$.

136. Problems often result in equations in which parentheses may be used.

EXAMPLES

1. Express by an equation: 2% of $(100-x)$ dollars equals $1.40.

2. Express by an equation: $\frac{4}{5}$ of the quantity $75-2x$ equals 44.

3. The difference between twice a number and the number less 10 is 22. Express this fact by an equation.

4. A man has $500 of which x dollars is in the bank drawing interest at 4%, and the remainder is lent at 6%. Express the annual interest received on the $500.

5. The amount of $100 for one year at 5% simple interest is $(1+.05)\,100$ dollars. Express the amount of x dollars at r% for 1 yr.; for 6 yr.

WRITTEN EXERCISES

Solve and test:

1. A man had $500, of which he invested x dollars at 4% per annum, and lent the remainder at 6% per annum. His annual interest was $28. Find how many dollars were invested at 4% and how many were lent at 6%.

2. Find the principal that will amount to $32.70 at the rate of $4\frac{1}{2}$% per annum for two years.

3. The amount of a certain principal at 4 % simple interest for $2\frac{1}{2}$ years is $220. What is the principal?

4. The amount of a certain principal at $5\frac{1}{2}$ % simple interest for $2\frac{1}{2}$ years is $284.375. What is the principal?

5. $100 is divided into two parts, of which one is x dollars. What is the other? If the first part exceeds the second by $10, find each part.

6. $600 is divided into two parts, one of which is $2x$. Write an expression for the other. The first part equals $\frac{1}{2}$ the second. Find each part.

7. The total amount of insurance in force in New York City and Buffalo in a recent year was $2,700,000,000; Buffalo had $\frac{2}{25}$ as much as New York. How much had each?

Solve Exercises 8–12 by equations requiring the use of the parenthesis: Then solve each by an equation not requiring a parenthesis:

8. Japan recently gave American manufacturers an order for 2000 cars and locomotives; it consisted of 19 times as many cars as locomotives. How many of each were ordered?

9. The amount of condensed milk produced by New York and Illinois is $\frac{3}{4}$ of that produced by the rest of the country; the total amount produced in the country annually is about 154 million pounds. How many pounds are produced by the two states together?

10. The United States produces 3 times as much cotton as the rest of the world; the total cotton production in a recent year was 14 million bales. What was the number of bales produced by the United States?

11. Mississippi and Texas together produced 4 million bales; Texas produced $1\frac{2}{3}$ times as much as Mississippi. How many bales did each produce?

12. The United States consumes $\frac{2}{5}$ as much cotton as does the rest of the world. How many bales is this, when the whole world consumes 14 million bales annually?

137. The use of the parenthesis often makes it possible to reduce the number of unknowns.

EXAMPLE

The sum of two numbers is 20, and 3 times one of them less 4 times the other is 4. Find the numbers.

This problem suggests at once two unknowns, but it can readily be solved by the use of one unknown, and practice in this work is good training in mathematics.

1. Let x be one of the required numbers.
2. Then, $20 - x$ is the other.
3. $3x - 4(20 - x) = 4$, by the conditions of the problem.
4. Therefore, $7x - 80 = 4$, and $x = 12$.
5. Then, $20 - x = 20 - 12$, or 8, and the numbers are 12 and 8.

The use of the parenthesis in step (3) takes the place of a second unknown.

WRITTEN EXERCISES

Solve and test:

1. The sum of two numbers is 25, and twice one of them plus 3 times the other is 60. What are the numbers?

2. The sum of two numbers is 30. $\frac{1}{3}$ of one of them less $\frac{1}{4}$ of the other is 3. What are the numbers?

3. The sum of two numbers is 38. One of them less $\frac{3}{8}$ of the other is 13. What are the numbers?

4. The sum of 2 numbers is 20. 2 times the larger number less 3 times the smaller is 5. Find the numbers.

5. The sum of two numbers is 42. When the larger number is diminished by 5, $\frac{1}{2}$ the result is the smaller number less 7. Find the numbers.

6. The product of two consecutive whole numbers diminished by the square of the smaller is 29. Find the numbers.

7. The product of two consecutive whole numbers less the square of the smaller is 49. Find the numbers.

8. The difference between two numbers is 2, and their product diminished by the square of the larger is -16. Find the numbers.

138. Certain problems of measurement are best solved by equations requiring the use of the parenthesis.

EXAMPLES

1. The inside measurements of a rectangular garden are 20 ft. by 30 ft. The outer margin is $1\frac{1}{2}$ ft. wide. What are the outside measurements of the garden? Express its area. If the margin were x ft. wide, express the area of the garden.

2. Express the outside measurements of a rectangular garden consisting of a walk $2a$ ft. wide around an inner plot 50 ft. by 70 ft. Express its area.

3. Express the difference between the area of a square x ft. on a side and the area of a square $(x + 1)$ ft. on a side.

4. Express the difference between the area of a square $(x + 2)$ ft. on a side, and the area of a square formed by diminishing this length by 5 ft.

WRITTEN EXERCISES

Solve and test:

1. The difference between the area of a square x ft. on a side and the area of a square $(x + 1)$ ft. on a side is 13 sq. ft. Find the side of the first square.

2. The difference between the area of a square $(x + 2)$ ft. on a side, and the area of a square formed by diminishing this length by 5 ft. is 75 sq. ft. Find the side of each square.

3. The inside measurements of a picture frame are 10 in. by 14 in. The width of the frame is x in. What are its outside measurements? If the area of the frame less $4x^2$ is 120 sq. in., what is its width?

4. The area of a square of side x equals the area of a rectangle, one of whose sides is $x - 6$, and the other $x + 12$. Find x.

5. The area of a square equals the area of a rectangle, one of whose sides exceeds the side of the square by 10 in., and the other is less than the side of the square by 6 in. Find the dimensions of each figure.

Problems of Motion:

1. If an electric car moves at the rate of $1\frac{1}{2}$ blocks per minute, how far will it move in 4 min.? If it moves at the rate of x miles per hour, how far will it move in $(m+1)$ hr.?

2. Let d equal the distance traveled in t hr. at the rate of r mi. per hour. Express d in terms of r and t.

3. Solve the equation $d = rt$ for t; solve it for r.

4. In $d = rt$, find d when $t = a + 1$ and $r = b - 1$. Find r when $d = x^2 - y^2$ and $t = x - y$.

5. $d = rt$ is the equation for distance when a body so moves that its rate may be taken as uniform. If sound travels 1100 ft. per second, how far away is a gun when the report of firing is heard $3\frac{1}{2}$ sec. after it occurred?

6. A fort is 10 mi. away. According to Exercise 5, how long after firing will it be before the report is heard?

7. If A travels x mi. an hour and B follows A at the rate of y mi. an hour, going faster than A, express the number of miles that B gains on A per hour. Express how long it will take B to gain d mi.

8. If A travels x mi. an hour and has c hr. the start of B, how far ahead is he when B starts? According to Exercise 7, in how many hours will B overtake A?

9. An automobile leaves city A at 7 A.M., going 20 mi. an hour. A motor cycle follows 2 hr. later, going 25 mi. an hour. How far ahead is the automobile at 9 A.M.? How far from A will the motor cycle overtake the automobile?

10. A steamer leaves its dock and travels 16 mi. per hour. It is followed 30 min. later by a motor boat traveling 20 mi. per hour. How far will they be from the dock when the motor boat overtakes the steamer?

11. Two cyclists A and B start at the same time from M and P respectively, 100 mi. apart, and travel toward each other, A at the rate of 15 mi. per hour and B at the rate of 20 mi. per hour. How many hours after starting do they meet, and how far from M?

REVIEW

WRITTEN EXERCISES

1. If C has x dollars and B has half as much, express what they both have. If this amount is \$75, write and solve an equation that will find what each has.

2. Express the area of a rectangle whose length is n ft. and whose width is 4 ft. If the area is 48 sq. ft., find n.

3 The width of a rectangle is x ft. and its length is twice the width. Express its perimeter. If the perimeter is 24 ft., find the length and width of the rectangle.

4. Express the bank discount on m dollars for 4 mo. at 6%. If the discount is \$3, find m.

5. Express the interest on s dollars at 6% for 3 yr. If this interest is \$36, find s.

6. Express the area of a triangle whose base is 6 in. and whose altitude is a in. If its area is 24 sq. in., find a.

7. A merchant marked an article d dollars and sold it at a 10% discount. Express the selling price. If the article brought \$7.20, find d.

8. In how many years will \$300 yield \$108 at 6% interest?

9. The perimeter of a rectangle is 30 ft. If the length of the base is twice the altitude, find the area of the rectangle.

10. A certain number plus twice the same number is 51. Find the number.

11. What number added to 3 times itself equals 64?

12. Divide the number 21 into three such parts that the first is twice the second and the second is twice the third.

13. If a certain number is multiplied by 12, the product is 168. Find the number.

14. A man sold a quantity of wood for \$49, half of it at \$3 a cord and the other half at \$4 a cord. How many cords of wood did he sell?

15. The sum of the ages of a father and son is 42 yr. and the father is five times as old as the son. What is the age of each?

16. Two men start from the same place and travel in opposite directions, one 35 mi. a day and the other 25 mi. a day. In how many days will they be 360 mi. apart?

17. Two men start from the same place and travel in the same direction, one 35 mi. a day and the other 25 mi. a day. In how many days will they be 360 mi. apart?

18. A, whose horse travels at the rate of 10 mi. an hour, starts 2 hr. after B from the same place. If B's horse travels at the rate of 8 mi. an hour, how many miles must A drive in the same direction to overtake B?

SUGGESTION. Let x equal the number of hours traveled by A before he overtakes B.

19. A flag pole 105 ft. high was broken so that the length of the part broken off was six times the length of the part left standing. Find the length of each part.

CHAPTER XI

TYPE PRODUCTS

139. Certain products are specially important because they serve as types or models for other multiplications. They apply to positive and negative numbers alike.

140. Type I: $x(y + z) = xy + xz.$

Type II: $x(y - z) = xy - xz.$

For example:

$$a(b + c) = ab + ac.$$
$$5\,x(3 - y) = 15\,x - 5\,xy.$$
$$2\,a^2(a - 5\,b) = 2\,a^3 - 10\,a^2b.$$
$$-3\,ab(c^2 - 4\,ad + b) = -3\,abc^2 + 12\,a^2bd - 3\,ab^2.$$

WRITTEN EXERCISES

Multiply:

1. $-x(x + y)$.
2. $c(a - b)$.
3. $a(t + t^2)$.
4. $cx(w + z)$.
5. $-y(x - y)$.
6. $t(u + \frac{1}{2}\,at)$.
7. $4\,ab(a + 2\,b)$.
8. $5\,xy(x^2 - y^2)$.
9. $pq(m - n)$.
10. $-2\,x(3\,x^2 - 2\,xy)$.
11. $-3\,a^xb^yx(a^2 - b^2)$.
12. $\frac{2}{3}\,xy(\frac{1}{2}\,x^2y^2 - 1)$.
13. $-4\,x^2(3\,x - 2\,y)$.
14. $2\,m^2(m - n^2)$.
15. $3\,y(4\,x - y)$.
16. $(5\,a^2 - 4\,b^2)(-a^2b^2)$.
17. $(5\,x - acy)(-acxy)$.
18. $(3\,a^2x - 8\,ax^3)(-3\,a^3x)$.
19. $(6\,am^p + 2\,bn^q)(-6\,m^2n)$
20. $(9\,a^3b^2 - 3\,cd^2)(-abcd)$.
21. $x(y + z + w)$.
22. $-3\,ab(a^2 - b^2 + c^2)$.
23. $\pi(r_1^2 + r_2^2 + r_1r_2)$.

141. Type III: $(x+y)^2 = x^2 + 2xy + y^2$.

In words:

The square of the sum of two numbers is the square of the first, plus twice the product of the first and second, plus the square of the second.

142. Type IV: $(x-y)^2 = x^2 - 2xy + y^2$.

In words:

The square of the difference of two numbers is the square of the first, minus twice the product of the first and the second, plus the square of the second.

For example:

$$(a+b^2)^2 = a^2 + 2ab^2 + b^4.$$

$$\begin{aligned}(2a-b)^2 &= (2a)^2 - 2(2a)b + b^2\\ &= 4a^2 - 4ab + b^2.\end{aligned}$$

$$\begin{aligned}23^2 &= (20+3)^2 = 20^2 + 2\cdot 20\cdot 3 + 3^2\\ &= 400 + 120 + 9 = 529.\end{aligned}$$

Evidently the above types include expressions either of the form $(ax+b)^2$ or $(ax-b)^2$.

WRITTEN EXERCISES

Square as indicated:

1. $(n+u)^2$.
2. $(x^2+y^2)^2$.
3. $(a+3b)^2$.
4. $(m+2n)^2$.
5. $(3x+2y)^2$.
6. $(a+2)^2$.
7. $(a^2+1)^2$.
8. $(a-b)^2$.
9. $(x-1)^2$.
10. $(2x-1)^2$.
11. $(x-\frac{1}{2})^2$.
12. $(t+w)^2$.
13. $(x+1)^2$.
14. $(2x+1)^2$.
15. $(2x^2+1)^2$.
16. $(2x^2+3y^2)^2$.
17. 25^2 or $(20+5)^2$.
18. 41^2.
19. 82^2.
20. 76^2 or $(80-4)^2$.
21. $(a^2-b^2)^2$.
22. $(3a-2b)^2$.
23. $(t-u)^2$.
24. $(abc-1)^2$.
25. $(mn+u^2)^2$.
26. $(2ab+bc)^2$.
27. $(abc+1)^2$.
28. 33^2.
29. 52^2.
30. 91^2.
31. 17^2.
32. 97^2.
33. 46^2.
34. 89^2.
35. 11^2.
36. 36^2.

143. Trinomials may also be squared by Types III and IV

For example:

$$(a+b+c)^2 = (\overline{a+b}+c)^2 = (a+b)^2 + 2(a+b)c + c^2$$
$$= a^2 + 2\,ab + b^2 + 2\,ca + 2\,bc + c^2$$
$$= a^2 + b^2 + c^2 + 2\,ab + 2\,bc + 2\,ca.$$
$$(2\,x - y + z)^2 = (2\,x - y)^2 + 2(2\,x - y)z + z^2$$
$$= 4\,x^2 - 4\,xy + y^2 + 4\,xz - 2\,yz + z^2.$$

In words:

The square of a polynomial is the sum of the squares of each of its terms and twice the product of every two.

WRITTEN EXERCISES

Square the trinomials as indicated:

1. $(\overline{a+b} - c)^2$.
2. $(2\,a - \overline{b+c})^2$.
3. $(a - \overline{3\,b - c})^2$.
4. $(x - y + z)^2$.
5. $(x + w - 2\,z^3)^2$.
6. $(\frac{1}{2}\,a - \frac{1}{8}\,b + \frac{1}{5}\,c)^2$.
7. $(mn + pq + rs)^2$.
8. $(1 - 6\,y + y^2)^2$.
9. $(m^2 + mp - q^2)^2$.

144. Type V: $(x + y)(x - y) = x^2 - y^2$.

In words:

The product of the sum and the difference of two numbers is the difference of their squares.

For example:

$$(a+b)(a-b) = a^2 - b^2.$$
$$(2\,a+b)(2\,a-b) = (2\,a)^2 - b^2 = 4\,a^2 - b^2.$$
$$(m^2+n^2)(m^2-n^2) = (m^2)^2 - (n^2)^2 = m^4 - n^4.$$

Evidently the above type includes expressions of the form $(ax + b)(ax - b)$.

ORAL EXERCISES

Multiply:

1. $(m - n)(m + n)$.
2. $(a - x)(a + x)$.
3. $(p - q)(p + q)$.
4. $(t + u)t(-u)$.
5. $(x - 1)(x + 1)$.
6. $(x - 2)(x + 2)$.
7. $(x + b)(x - b)$.
8. $(2\,x - 1)(2\,x + 1)$.
9. $(2\,x - y)(2\,x + y)$.
10. $(1 + x^3)(1 - x^3)$.
11. $(ax + by)(ax - by)$.
12. $(2\,x - 3\,y)(2\,x + 3\,y)$.
13. $(2\,a^2 + 3)(2\,a^2 - 3)$.
14. $(a^2 + b)(a^2 - b)$.
15. $(a^2 - 3\,ax)(a^2 + 3\,ax)$.

16. $(ax - x^2)(ax + x^2)$.
17. $(\frac{1}{2}x - \frac{1}{3}y)(\frac{1}{2}x + \frac{1}{3}y)$.
18. $(a - x)(a + x)(a^2 + x^2)$.
19. $(a - x)(a + x)(a^2 + x^2)(a^4 + x^4)$.
20. $(1 - r)(1 + r)(1 + r^2)(1 + r^4)(1 + r^8)$.
21. $(1 - r)(1 + r)(1 + r^2)(1 + r^4)(1 + r^8)(1 + r^{16})$.

145. Two numbers, one greater than a multiple of 10, and the other less than this multiple by the same amount, may be multiplied according to Type V.

Thus, $93 \cdot 87 = (90 + 3)(90 - 3) = 90^2 - 3^2 = 8100 - 9 = 8091$.

WRITTEN EXERCISES

1. $31 \cdot 29 = (30 + 1)(30 - 1) = ?$
2. $42 \cdot 38 = (40 + 2)(40 - 2) = ?$
3. $35 \cdot 45 = ?$
4. $57 \cdot 63 = ?$
5. $21 \cdot 19 = ?$
6. $32 \cdot 28 = ?$
7. $29 \cdot 31 = ?$
8. $66 \cdot 54 = ?$
9. $44 \cdot 36 = ?$
10. $91 \cdot 89 = ?$
11. $53 \cdot 47 = ?$
12. $16 \cdot 24 = ?$
13. $99 \cdot 101 = ?$
14. $98 \cdot 102 = ?$
15. $90 \cdot 110 = ?$
16. $127 \cdot 113 = ?$

17. What is the cost of 21 doz. eggs at 19 ¢ a dozen?

18. What is the cost of 28 lb. of butter at 32 ¢ a pound?

19. How many oranges in 146 crates of 154 oranges each?

20. What is the area of a rectangle whose dimensions are 62 ft. and 58 ft.?

21. How far does a train travel in 37 hr. at the rate of 43 mi. per hour?

146. Type VI: $(x + a)(x + b) = x^2 + (a + b)x + ab$.

For example:

$(x + 5)(x + 3) = x^2 + 8x + 15$.
$(x - 3)(x + 7) = x^2 + 4x - 21$.
$(3x + c)(3x - d) = (3x)^2 + (c - d)3x - cd = 9x^2 + 3(c - d)x - cd$
$91 \cdot 87 = (100 - 9)(100 - 13) = 100^2 - 22 \cdot 100 + 9 \cdot 13$
$= 10000 - 2200 + 117 = 7917$.

Evidently this type includes expressions of the form

$$(ax + b)(ax + c).$$

ORAL EXERCISES

State the products:

1. $(x+2)(x+5)$.
2. $(a+3)(a+6)$.
3. $(b-5)(b+2)$.
4. $(4x+7)(4x-5)$.
5. $(5+m)(5+r)$.
6. $(4g-5)(4g-9)$.
7. $(p+q)(p+2q)$.
8. $(7-x)(7-y)$.
9. $(ab+c)(ab+d)$.
10. $(2a-7b)(2a+8b)$.

WRITTEN EXERCISES

Find the products:

1. $93 \cdot 95$.
2. $197 \cdot 191$.
3. $(x+14)(x-19)$.
4. $(2z-3a)(2z+5a)$.
5. $993 \cdot 985$.
6. ✓ $(-3x+11)(-3x-6)$.
7. $(15x+23)(15x-25)$.
8. $(4w+a)(4w-2a)$.

147. Type VII. Expressions of the form $(ax+b)(cx+d)$ have the product $acx^2+(bc+ad)x+bd$.

EXAMPLES

$$\begin{aligned}\overparen{(3a + \overparen{5)(4}a + 7)} &= 12a^2 + (20+21)a + 35\\ &= 12a^2 + 41a + 35.\end{aligned}$$

$$\overparen{(ax - \overparen{b)(c}x - d)} = acx^2 - (bc+ad)x + bd.$$

The coefficient of the middle term is the sum of the products indicated by the curved lines above the given expression.

Actual multiplication is quite as simple, but practice in forming products as above indicated is a good preparation for factoring expressions of this type.

ORAL EXERCISES

Find the products:

1. $(2a+5)(3a-1)$.
2. $(a-5)(4a+2)$.
3. $(3a+1)(a-3)$.
4. $(x+4)(3x-5)$.
5. $(2x-a)(x-b)$.
6. $(3xy-1)(5xy+2)$.
7. $(x^n-2)(3x^n-4)$.
8. $(2x^n-a)(x^n+b)$.

WRITTEN EXERCISES

Multiply:

1. $(ay+3)(by+5)$.
2. $(2\,ay+4)(3\,ay-5)$.
3. $(3\,ax+b)(ax-c)$.
4. $(15\,b+2\,c)(5\,b-10\,c)$.
5. $(40\,b^2c-1)(5\,b^2c+d)$.
6. $(2\,x^p-ab)(5\,x^p+bc)$.
7. $(mx^n-y^p)(px^n+2\,y^p)$.
8. $(aby^r-c)(ab^2y^r-d)$.
9. $(cdx^p-y^2)(acx^p-dy^2)$.
10. $(xy^n-mn)(zy^n+pq)$.

148. Type VIII. $(x+y)^3=x^3+3\,x^2y+3\,xy^2+y^3$.

Since by actual multiplication

$$(x+y)^3=x^3+3\,x^2y+3\,xy^2+y^3,$$

therefore, $$(x+a)^3=(\ )^3+3(\ )^2a+3(\ )a^2+(\ )^3.$$

Also, $$\begin{aligned}(a+b^2)^3&=(\ )^3+3(\ )b^2+3(\ )(b^2)^2+(b^2)^3\\&=(\ )+3(\ )+3(\ )+(\ ).\end{aligned}$$

Since by actual multiplication

$$(x-y)^3=x^3-3\,x^2y+3\,xy^2-y^3,$$

therefore, $$(m-n)^3=(\ )^3-3(\ )+3(\ )-(\ )^3.$$

Similarly, $$\begin{aligned}(ab-c)^3&=(ab)^3-3(ab)^2c+3\,abc^2-c^3\\&=(\ )-3(\ )+3\,abc^2-c^3.\end{aligned}$$

Similarly, $$\begin{aligned}(\overline{a-b}+c)^3&=(a-b)^3+3(a-b)^2c+3(a-b)c^2+c^3\\&=(\ )+3(\ )+3(\ )+(\ ).\end{aligned}$$

WRITTEN EXERCISES

Expand by Type VIII:

1. $(m+n)^3$.
2. $(p-q)^3$.
3. $(a-x)^3$.
4. $(2\,a+x)^3$.
5. $(a^2+b^2)^3$.
6. $(m^2-n^2)^3$.
7. $(a+2\,b)^3$.
8. $(a-3\,c)^3$.
9. $(2\,a-3\,b)^3$.
10. $a(a+b)^3$.
11. $ax(x-y)^3$.
12. $(ab+cd)^3$.
13. $(x+1)^3$.
14. $(3\,x-1)^3$.
15. $(y^2-1)^3$.
16. $(a^m+b^m)^3$.
17. $(x^n-y^n)^3$.
18. $(\frac{1}{2}\,x-\frac{1}{3}\,y)^3$.
19. $(x^n-y^m)^3$.
20. $(a^2b+ab^2)^3$.
21. $(\overline{m+n}-p)^3$.
22. $(m+\overline{n-p})^3$.
23. $(a+b+c)^3$.
24. $(2\,a+b+c)^3$.

REVIEW

ORAL EXERCISES

State the products of:

1. $x(x+3)$.
2. $x(a+b)$.
3. $4r(r-2)$.
4. $3a(3b-4c)$.
5. $b(a-c)$.
6. $p(m-p)$.
7. $(3+t)(3-t)$.
8. $(a-y)(a+y)$.
9. $(7-x)(7+x)$.

State the square as indicated:

10. $(a+x)^2$.
11. $(1-x)^2$.
12. $(7-y)^2$.

WRITTEN EXERCISES

Write the products:

1. $(2a-1)^3$.
2. $(10+3)^3$.
3. $(5x-y)(5x+y)$.
4. $(ax+8)(ax-8)$.
5. $(ax^m-1)(ax^m-3)$.
6. $(20+1)(20-1)$.
7. $(3a+x)(3a-x)$.
8. $(3a-5)(4a-6)$.
9. $(6a-12)(7a+15)$.
10. $(2a-3b)(7a+5b)$.

Remove the parentheses and unite terms where possible:

11. $(3x-1)^2+2(4x+3)^2$.
12. $5(7y-4)-(4y+3)^2$.
13. $(79)^2+(92)^2$.
14. $(ab-c)^2+(ab+c)^2$.
15. $(x-3y)(x+3y)+(x-5y)^2$.
16. $3x(7y-4)-(2x+3y)^2$.
17. $133 \cdot 127$ or $(130+3)(130-3)$.
18. $4a(b-1)+2(3a-b)^2$.

Find the product:

19. $(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})$.

20. Show by multiplying that

$$s(s-a)(b+c)+a(s-b)(s-c)-2bcs$$

is identical with

$$s(s-b)(a+c)-b(a-s)(s-c)-2acs.$$

CHAPTER XII

FACTORING

149. The factoring of algebraic expressions depends largely upon a knowledge of products, and for this reason the special products most used in factoring were brought together and emphasized in the previous chapter.

I. ROOTS

150. PREPARATORY.

1. $(+2)(+2)=?$ $\qquad$ $(-2)(-2)=?$

2. State a number which taken twice as a factor produces 4. State another number which taken twice as a factor produces 4.

3. According to Exercise 2, how many square roots has 4? What are they?

4. Similarly, name the square roots of 9; 16; 25; 36.

151. Signs of Square Roots. Every number has two square roots which differ only in their signs.

Thus, $\sqrt{4}=+2$ or -2; because $(+2)(+2)=4$, and $(-2)(-2)=4$.

$\sqrt{a^2}=+a$ or $-a$; because $(+a)(+a)=a^2$, and $(-a)(-a)=a^2$.

It should be noticed that, although either square root taken twice as a factor produces the given number, the product of the two square roots is not equal to the given number.

152. The sign ± is used to denote that a number may be taken either positively or negatively.

Thus, $\sqrt{4}=+2$ or -2 is written $\sqrt{4}=\pm 2$.
Also, $\sqrt{a^2}=+a$ or $-a$ is written $\sqrt{a^2}=\pm a$.

The positive square root of a number is called the **principal square root.**

ORAL EXERCISES

State the two square roots of each number:

1. 25.
2. 49.
3. 121.
4. 196.
5. 225.
6. 81.
7. 625.
8. 169.
9. 144.
10. 36.
11. a^2b^2.
12. $36a^2b^2$.
13. $49a^2x^4$.
14. $25p^2q^6$.
15. $\frac{1}{4}g^2t^2$.
16. $\frac{9}{16}m^2n^8$.

153. The square roots of a monomial may often be found by factoring.

EXAMPLE

Find the square root of $576m^2n^4$.

By trial, $576 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 2^3 \cdot 3 \cdot 3$,

and $m^2n^4 = mn^2 \cdot mn^2$.

$$\therefore\ \sqrt{576m^2n^4} = \pm 2^3 \cdot 3 \cdot m \cdot n^2$$
$$= \pm 24mn^2.$$

WRITTEN EXERCISES

Find by factoring:

1. $\sqrt{625m^2n^2}$.
2. $\sqrt{11025x^6}$.
3. $\sqrt{256a^4b^4}$.
4. $\sqrt{225x^6y^6}$.
5. $\sqrt{3025x^2y^2}$.
6. $\sqrt{961a^6b^8}$.
7. $\sqrt{3136m^4n^8}$.
8. $\sqrt{169(a+b)^2}$.
9. $\sqrt{9216(x-y)^4}$.

154. Cube Root. A **cube root** of a given number is a number whose third power (or cube) equals the given number.

For example:

4 is a cube root of 64 because $4 \cdot 4 \cdot 4 = 64$.
bc is a cube root of b^3c^3 because $bc \cdot bc \cdot bc = b^3c^3$.
$-3ab$ is a cube root of $-27a^3b^3$.
$2(a+b)$ is a cube root of $8(a+b)^3$.
$-a(b+c)$ is a cube root of $-a^3(b+c)^3$.

The sign of the cube root is the same as the sign of the power.

155. Evolution. The finding of roots of numbers is called **evolution.**

ORAL EXERCISES

Name a cube root of:

1. a^6.
2. $27\,a^3$.
3. $-8\,a^3b^3$.
4. $8\,x^3y^3z^3$.
5. $27\,x^3y^3m^6$.
6. $8(x+y)^3$.
7. $-a^3(b+c)^3$.
8. $8\,m^3(m+n)^3$.
9. $125(p-q)^3$.
10. $x^6y^6(b-c)^3$.
11. $(a+b)^3(x+y)^3$.
12. $(x-y)^6(x+y)^3$.

II. MONOMIAL FACTORS

156. When every term of a polynomial contains a common factor, that factor may usually be found by inspection. (Sec. 140.)

For example:

$3\,ab$ is a factor of $3\,abx-6\,aby-9\,abz$, for $3\,abx-6\,aby-9\,abz = 3\,ab\cdot x-3\,ab\cdot 2\,y-3\,ab\cdot 3\,z$.

ORAL EXERCISES

State the monomial factor of each expression:

1. $ab+ac+ad$.
2. $ab+bc+b$.
3. $2\,ax+2\,ay+2\,az$.
4. $m^2+m^2y+m^2z$.
5. $3\,mx-6\,my-9\,mz$.
6. $5\,ab+10\,a^2b^2-5\,abc$.
7. $2\,a(x-y)+2\,axy$.
8. $a^2b^2-3\,ab^3+5\,a^3b$.
9. $10^{n+3}+10^{n+5}+10^n$.
10. $7\,acx^2-3\,bcx^3-2\,c^2x^4$.
11. $12\,am^2+4\,ar^2-6\,a^2rm$.
12. $6\,x^2y+10\,xy^2-18\,x^2y^2$.
13. $2\,ax^3+12\,a^2x^2-24\,abx^2$.
14. $\dfrac{m^2p^2}{5\,a}+\dfrac{3\,mp^2}{2\,a^2}-\dfrac{7\,m^2p}{10\,a^3}$.
15. $6\,a^2b^2c^2-2\,a^3bc-2\,abc$.
16. $3\,ay^2+6\,a^2y-9\,a^3y^3$.
17. $x^3-6\,x^2+12\,x$.
18. $x^2y^2+xy^2+xyz$.
19. $3\,x^3y-6\,x^2y^2+9\,x^2y^3$.
20. $4\,a^2x-8\,a^2y-6\,a^2b^2$.
21. $3\,a^4b^3-3\,a^5-15\,a^3b^2$.
22. $15\,a^2x-10\,a^2y+5\,a^2z$.
23. $a^{2n+4}b^3-a^{3n+4}b^7$.
24. $4\,px^2-20\,p^2x^3-10\,pxy$
25. $36\,q^3+108\,q^5-18\,pq^4$.
26. $5^{a+1}-5^{a+5}+3\cdot 5^a$.
27. $3x(2y-1)-6x^2y-9xyz$.
28. $\dfrac{3\,xy^2}{2\,p^2}-\dfrac{6\,x^2y}{p^3}+\dfrac{12\,xyz}{5\,p^5}$.

WRITTEN EXERCISES

1–28. Write the other factor in Exercises 1–28 above.

✓ III. POLYNOMIAL FACTORS

157. An expression may have a binomial or other polynomial factor that can readily be found by inspection.

For example:

1. $a + b$ is a factor of $(a + b)x + (a + b)y$.
2. And $x + y - z$ is a factor of $(x + y - z)ab - 3(x + y - z)cd$.
3. $3x^3 - 5x^2 + 3x - 5 = x^2(3x - 5) + 1(3x - 5) = (3x - 5)(x^2 + 1)$.

Note that $3x - 5$ divides each term of the middle, or grouped, expression giving the quotients x^2 and $+1$.

ORAL EXERCISES

State a factor of:

1. $(a+1)x-(a+1)y$.
2. $(a+x)x-(a+x)y$.
3. $a(b+c)x^2-a(b+c)y^2$.
4. $(m+n+p)ab+(m+n+p)cd$.
5. $(x+y)^2-(x+y)$.
6. $(a+1)^3+(a+1)^2+(a+1)$.

Supply the blanks:

7. $ax+ay+bx+by=(\quad)(x+y)+(\quad)(x+y)$
$=[(\quad)+(\quad)](x+y)$.

8. $ax+bx-ay-by=(\quad)(a+b)-(\quad)(a+b)$
$=[(\quad)-(\quad)](a+b)$.

9. $2ax^2-4ax+3x-6=(\quad)(x-2)+(\quad)(x-2)$
$=[(\quad)+(\quad)](x-2)$.

10. $6a+3b+9c+2ax+bx+3cx=(\quad)(2a+b+3c)$
$+(\quad)(2a+b+3c)=[(\quad)+(\quad)](2a+b+3c)$.

WRITTEN EXERCISES

Factor:

1. x^3+x^2+x+1.
2. x^3-2y-x^2y+2x.
3. $ax-ay+bx-by$.
4. $ax+3a+bx+3b$.
5. $a^2+ab-ac-bc$.
6. $ax+x-ay-y$.
7. $x-a+(x-a)^2$.
8. $5h^3-4h^2+10h-8$.

9. $6m^3 + 4m^2 - 9m - 6.$

10. $xy - by - b + x.$

11. $x(z-a)^2 - y(a-z).$

12. $a(x+1)^2 + 3x + 3.$

13. $4ax + bx - 4ay - by.$

14. $4a^3 + a^2 - 4a - 1.$

15. $x^3 - 4x^2 + 2x - 8.$

16. $x^2 - x - 2x + 2.$

17. $(a+b+c)^2 + ax + bx + cx.$

18. $(m+r)^2x^2 + my + ry - m - r.$

19. $x(p-y) + ap - ay.$

20. $ax + ay - x(x+y)$

IV. SQUARES OF BINOMIALS

158. Since $(x \pm y)^2 = x^2 \pm 2xy + y^2$, a trinomial is the square of a binomial, if one term is twice the product of the square roots of the other two, but not otherwise. (Secs. 141, 142.)

For example:

$$a^2 + 14a + 49 = a^2 + 2 \cdot 7a + 7^2 = (a+7)^2.$$

Here $14a$ is twice the product of $\sqrt{a^2}$ and $\sqrt{49}$.

$$25m^2 - 30m + 9 = (5m)^2 - 2 \cdot 3 \cdot 5m + 3^2 = (5m-3)^2.$$

$$16a^6 - 8a^3 + 1 = (4a^3)^2 - 2 \cdot 4a^3 \cdot 1 + (1)^2 = (4a^3 - 1)^2.$$

Test by squaring.

WRITTEN EXERCISES

Factor:

1. $x^2 + 2ax + a^2.$

2. $x^2 - 2mx + m^2.$

3. $4x^2 - 4x + 1.$

4. $9x^2 - 12x + 4.$

5. $x^2y^2 + 2xy + 1.$

6. $a^{2x}b^{2y} + 2a^xb^yc + c^2.$

7. $a^2b^2 + 2abmn + m^2n^2.$

8. $a^2x^2 - 8ax + 16.$

9. $p^2 - 4pq + 4q^2.$

10. $4p^2x^2 - 4px + 1.$

11. $25a^2b^2 - 10abc + c^2.$

12. $9x^{2a} - 12x^a + 4.$

13. $x^2y^4 - 2xy^2 + 1.$

✓14. $\frac{4}{9}a^2 - \frac{2}{3}a + \frac{1}{4}.$

✓15. $\frac{1}{4}x^2 + \frac{1}{2}xy + \frac{1}{4}y^2.$

✓16. $p^2q^2 - \frac{2}{3}pq + \frac{1}{9}.$

17. $1 - \frac{2}{x^a} + \frac{1}{x^{2a}}.$

18. $(1-p)^2 - 6(1-p) + 9.$

19. $x^2 - 2(a-b) + (a-b)^2.$

20. $4(a^2+1)^2 + 4(a^2+1) + 1.$

21. $\frac{4}{x^2} + 20 + 25x^2.$

22. $y^{2a}-4y^a+4$.

23. $a^4x^2-a^2x+\frac{1}{4}$.

24. $(x+y)^2+2(x+y)+1$.

25. $\dfrac{1}{x^2}+\dfrac{2}{xy}+\dfrac{1}{y^2}$.

26. $1+\dfrac{2}{x}+\dfrac{1}{x^2}$.

27. $(m-n)^2-2(m-n)x+x^2$.

28. $(a+b)^2-2(a+b)y+y^2$.

29. $9(p+1)^2-6(p+1)+1$.

30. $x^{2m}+4x^m+4+2(x^m+2)+1$.

31. $p^2q^2x^2y^2-6pqxy+9$.

32. $a^{2p}+144-24a^p$.

33. $p^{6n}+49-14p^{3n}$.

34. $49x^2+81y^2-126xy$.

35. $\dfrac{1}{m^4}+\dfrac{2}{m^2}+1$.

36. $\dfrac{x^2}{y^2}+2+\dfrac{y^2}{x^2}$.

37. $x^2+2xy+y^2+2(x+y)z+z^2$.

38. $a^2+2a(p-q)+p^2-2pq+q^2$.

REVIEW EXERCISES

Factor:

1. $ax+ab+ay^2$.

2. $1-2y+y^2$.

3. $3a^2x-6ay+9abxy$.

4. $(x+y)^2+a(x+y)$.

5. $16m^2-24mn+9n^2$.

6. $px+py+qx+qy$.

7. $\dfrac{4}{x^2}+\dfrac{4}{x}+1$.

8. $\dfrac{p^2}{q^2}-2+\dfrac{q^2}{p^2}$.

9. $\dfrac{1}{y^4}-\dfrac{2}{y^2}+1$.

10. $ax-bx-a+b$.

11. $(m-2)^2-3m+6$.

12. $16a^4-8a^2+1$.

13. $ap+bp-aq-bq$.

14. $4a^{2p}b^{4p}-4a^pb^{2p}+1$.

15. $\dfrac{6x^2}{y}-\dfrac{9x}{y^2}+\dfrac{12x^2}{5y^3}$.

16. $81x^2+121y^2-198xy$.

17. $x^{n+1}+x^{n+2}-x^n$.

18. $5a^{2n+1}b^p-15a^{3n-1}b^{p+1}$.

19. $mxy+nxy+pxy+mcd$
$+ncd+pcd$.

20. $4x+6y+8z+2ax+3ay$
$+4az$.

21. $p^2-2pq+q^2+2(p-q)r$
$+r^2$.

22. $y^{2p}-4y^p+4-2(y^p-2)$
$+1$.

23. $abx+aby-abz-3cdx$
$-3cdy+3cdz$.

V. THE DIFFERENCE OF TWO SQUARES

159. The factors of the difference of two squares are the sum and the difference of the numbers whose squares are given.

For example:

The factors of $a^2b^2 - 1$ are $ab + 1$, $ab - 1$, because

$$(ab + 1)(ab - 1) = a^2b^2 - 1.$$

The factors of $a^4 - 4c^2d^2$ are $a^2 + 2cd$, $a^2 - 2cd$, because

$$(a^2 + 2cd)(a^2 - 2cd) = a^4 - 4c^2d^2.$$

ORAL EXERCISES

Read and supply the blanks:

1. $t^2 - v^2 = (t - v)(\quad)$.
2. $a^{2x} - 4b^{2y} = (a^x - 2b^y)(\quad)$.
3. $4x^2 - y^2 = (2x + y)(\quad)$.
4. $9x^2 - 4y^2 = (3x - 2y)(\quad)$.
5. $a^2x^{2m} - y^{2n} = [ax^m - (\quad)][ax^m + (\quad)]$.
6. $25s^2 - 49t^2 = [5s + (\quad)][5s - (\quad)]$.

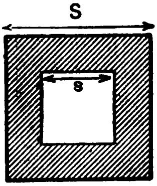
WRITTEN EXERCISES

Write the factors of:

1. $1 - y^2$.
2. $81u^2 - 64v^2$.
3. $121t^2 - 4$.
4. $1 - 144q^2$.
5. $x^2y^2 - 25$.
6. $144a^2b^2 - 49c^2$.
7. $4x^4 - y^4$.
8. $a^4 - 9b^4$.
9. $x^{10} - y^8$.
10. $y^4 - \frac{1}{4}$.
11. $92^4 - 1$.
12. $36x^4 - 49y^4$.
13. $26^2 - 24^2$.
14. $41^2 - 31^2$.
15. $32^2 - 28^2$.
16. $a^2 - 36b^4$.
17. $763^2 - 663^2$.
18. $36x^2y^2 - 169t^2u^2$.

19. Calculate the area of the shaded portion of this square, if

SIDE	(1)	(2)	(3)	(4)	(5)	(6)
$S =$	49	290	597	73	$6a$	$24q$
$s =$	45	280	497	27	$4a$	$14q$



Factor:

20. $a^2-b^2x^2$.
21. x^4-y^2.
22. $4\,a^2c-b^2c$.
23. $9\,x^2-16\,y^2$.
24. $25\,a^2b^2-1$.
25. $25\,a^2t^3-4\,b^2t^3$.
26. $49\,x^{2p+6}-1$.
27. $49\,a^2-4\,b^2$.
28. $1-a^2b^2c^2$.
29. $1-121\,x^2y^2$.
30. $1-x^4$.
31. $7\,a^2b^2-7\,c^2d^2$.
32. $16\,x^2y^2-4\,m^2n^2$.
33. ax^4-ay^4.
34. $x^6-4\,y^6$.
35. $c^5x^8-4\,c^5y^8$.
36. $81\,x^2y^2-9$.
37. $225\,a^4b^4-1$.
38. $25\,x^2y^4-36\,x^4y^2$.
39. $100a^2b^6-25a^6b^2$.
40. $\dfrac{a^2}{b^2}-1$.
41. $\dfrac{x^{2m}}{y^{2m}}-9$.
42. $16-\dfrac{p^{2n+4}}{q^{2n+4}}$.
43. $\dfrac{4\,x^2}{9\,y^2}-\dfrac{252^2}{49}$.
44. $\dfrac{16}{625\,x^2}-1$.

160. The terms of the given square may be polynomials, but the method of factoring is the same.

For example:

$$(a-b)^2-(b+c)^2=[(a-b)+(b+c)][(a-b)-(b+c)]$$
$$=(a+c)(a-2\,b-c).$$
$$(a^2+b)^2-(x^2+y)^2=(a^2+b+x^2+y)(a^2+b-x^2-y).$$

WRITTEN EXERCISES

Factor:

1. $(x+y)^2-(x-y)^2$.
2. $(a+b)^2-(a-b)^2$.
3. $(p+q)^2-(m+n)^2$.
4. $(a+b+c)^2-z^2$.
5. $(a-2\,b)^2-(3\,b+c)^2$.
6. $(a^2-1)^2-(b^2-1)^2$.
7. $a^2+2\,ab+b^2-c^2$.
8. $x^2-2\,xy+y^2-z^2$.
9. $4\,a^2-4\,a+1-9\,b^2$.
10. $16\,a^2b^2-a^2c^2-6\,ac-9$.
11. $p^2t^2-10\,pt+25-p^2+10\,pt-25\,t^2$.
12. $a^2-4\,ab+4\,b^2-9\,y^2$.
13. $4\,a^2+8\,ab+4b^2-1$.
14. $4-x^2-2\,xy-y^2$.
15. $4\,x^2-4\,a^2-4\,ab-b^2$.
16. $x^2-2\,xy+y^2-16\,c^2$.
17. $(3\,x-2\,y)^2-(2\,x+3\,y)^2$.
18. $(x^n-y^n)^2-(x^n+y^n)^2$.
19. $x^{2p}+2\,x^p+1-a^2b^2$.
20. $(m-2n)^2-36\,y^2$.
21. $(2\,x+y)^2-(3\,x-4)^2$.
22. $(3a-4\,b)^2-(5a+6\,b)^2$.
23. $(2\,x+y-z)^2-(5\,x+6y)^2$.
24. $9\,a^{2x}-6\,a^xb+b^2-9\,c^2$.
25. $(5\,z^m+2\,p-1)^2-x^{2a}y^{4p}$.

VI. FORMING THE DIFFERENCE OF TWO SQUARES

161. It is often possible to factor an expression by first making it the difference of two squares.

If the given expression can be made the square of a binomial by adding a square, it can be made the difference of two squares by adding and subtracting the same square.

For example:

$$a^4 + a^2b^2 + b^4 = a^4 + 2a^2b^2 + b^4 - a^2b^2 = (a^2 + b^2)^2 - a^2b^2$$
$$= (a^2 + b^2 + ab)(a^2 + b^2 - ab).$$
$$x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2 + 2)^2 - (2x)^2$$
$$= (x^2 + 2 - 2x)(x^2 + 2 + 2x).$$
$$16x^4 - x^2 + 1 = 16x^4 + 8x^2 + 1 - 9x^2 = (4x^2 + 1)^2 - 9x^2$$
$$= (4x^2 + 1)^2 - (3x)^2 = (4x^2 + 1 + 3x)(4x^2 + 1 - 3x)$$

TEST. $16 - 1 + 1 = 16 = (4 + 1 + 3)(4 + 1 - 3).$

WRITTEN EXERCISES

Express as a difference of two squares and factor:

1. $a^4 + 4.$
2. $x^4y^4 + 4.$
3. $a^8 + 64.$
4. $64 + b^4.$
5. $x^4 + 4 \cdot 10^{8n}.$
6. $x^{4n} + 2x^{2n} + 9.$
7. $x^4 - 6x^2y^2 + y^4.$
8. $x^4 + 3x^2y^2 + 4y^4$
9. $16a^4 + 4a^2 + 1.$
10. $4x^4y^4 + 3x^2y^2 + 1.$
11. $x^4 + 4y^4.$
12. $81p^4 + 9p^2 + 1.$
13. $x^4 + 25x^2y^2 + 625y^4.$
14. $16a^4b^4 + 4a^2b^2 + 1.$
15. $81a^4 + 225a^2b^2 + 625b^4.$
16. $625x^4 + 400x^2y^2 + 256y^4.$
17. $a^4 + 2a^2b^2 + 9b^4.$
18. $x^4 - 8x^2y^2 + 4y^4.$
19. $4a^4 - 16a^2b^2 + 9b^4.$
20. $x^2 + 2xy - 15y^2.$

REVIEW EXERCISES

Factor:

1. $3x^3 - 9x^2 + x - 3.$
2. $m^4 - m^2 - 5m - 5.$
3. $16x^4 + 4x^2 + 1.$
4. $4p^4 - 8p^2q^2 + q^4.$
5. $25a^2 - 40ab + 16b^2 - 9c^2.$
6. $81x^2 - y^2 - 4yz - 4z^2.$
7. $b(y - z)^2 + 5y - 5z.$
8. $(p + q - r)^2 + ap + aq - ar.$

9. $64 + y^4$.
10. $(m - n)^2 - 4x^4y^2$.
11. $p^4 + 4z^{4n}$.
12. $(x^a - b)^2 + 2x^a - 2b + 1$.
13. $(x + y)^2 + 2(x + y)z + z^2$.
14. $bx^2 + x^2 - by^2 - y^2$.
15. $(2x - y)^{n+1} - (2x - y)^n$.
16. $64r^4 - \frac{1}{16}$.
17. $36x^{2m+6} - 1$.
18. $\frac{x^2y^2}{p^2} - \frac{p^2}{m^2q^2}$.
19. $x^{2a} + 81 - 18x^a$.
20. $x^{2a}y^{2b} + 2x^ay^b + 1$.
21. $36m^2 + 49p^2 - 84mp$.
22. $15a^2xy + 25a^2x^2y - 10a^2xy^2$.
23. $\frac{3xy}{5} - \frac{x^2}{15} + \frac{4x^2z}{75}$.
24. $4x + 3y - z - 12ax - 9ay + 3az$.

VII. TYPE $x^2 + px + q$

162. $(x + a)(x + b) = x^2 + (a + b)x + ab$. Hence if a trinomial is the product of two factors like $x + a$ and $x + b$, the sum of a and b is the coefficient of the middle term, and their product, ab, is the third term.

The factors of such a trinomial are seen at once if a and b can be found by inspection.

For example:

In $x^2 + 5x + 6$, $5 = 2 + 3$, and $6 = 2 \cdot 3$.
$\therefore\ x^2 + 5x + 6 = x^2 + (2 + 3)x + 2 \cdot 3 = (x + 2)(x + 3)$.

In $x^2 - 7x + 6$, $6 = (-6)(-1)$, and $-7 = -6 + (-1)$.
$\therefore\ x^2 - 7x + 6 = x^2 + (-6 - 1), x + (-6)(-1) = (x - 6)(x - 1)$.

In $x^2 - 5x - 6$, $-6 = 6(-1)$, and $5 = 6 + (-1)$.
$\therefore\ x^2 + 5x - 6 = x^2 + (6 - 1)x + 6(-1) = (x + 6)(x - 1)$.

ORAL EXERCISES

Factor:

1. $a^2 + 3a + 2$.
2. $d^2 - 3d + 2$.
3. $a^2 - 5a + 4$.
4. $z^2 - 8z + 15$.
5. $m^2 - m - 12$.
6. $x^2 + 5x + 4$.
7. $y^2 - 6y + 5$.
8. $m^2 + 6m + 5$.
9. $w^2 + 4w - 21$.
10. $z^2 - 2z - 15$.
11. $x^2 + 8x + 7$.
12. $x^2 - 8x + 7$.
13. $a^2 - a - 2$.
14. $m^2 - 7m + 12$.
15. $w^2 - 4w - 21$.

163. Although the inspection method of Section 162 should be generally used, it may be helpful in some cases to write the various pairs of factors of the third term and then compare their sum with the coefficient of the middle term.

For example:

In $x^2 - 17x + 72$, the pairs of factors of $+72$ are

$$\begin{array}{cccccc} 72 & 36 & 24 & 18 & 12 & 9 \\ \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{6} & \underline{8} \end{array}$$

and the same pairs taken negatively.

Since the sum of the factors is negative, only the negative pairs need be examined, and by trial the pair -8, -9 is found to have the sum -17.

$$\therefore x^2 - 17x + 72 = (x-8)(x-9).$$

Likewise, in $x^2 - x - 56$, the factors of -56 are

$$\begin{array}{ccc} -28 & -14 & -8 \\ \underline{2} & \underline{4} & \underline{7} \end{array}$$

or the same numbers with the signs changed; but since the coefficient of x is negative, only those pairs need be examined in which the negative number is the larger.

By inspection, -8, 7 are found to have the sum -1.

$$\therefore x^2 - x - 56 = (x-8)(x+7).$$

Test by multiplication.

WRITTEN EXERCISES

Factor (use Section 163 for 7, 10, 12, 16):

1. $x^2 - x - 30$.
2. $x^2 + x - 30$.
3. $x^2 - x - 20$.
4. $x^2 + x - 20$.
5. $x^2 - 3x - 18$.
6. $x^2 + 3x - 18$.
7. $a^2 + a - 42$.
8. $a^2 - a - 12$.
9. $m^2 - \frac{5}{6}m + \frac{1}{6}$.
10. $x^2 - 7x - 18$.
11. $x^2 + 7x - 18$.
12. $x^2 + 17x + 60$.
13. $m^2 + 11m + 28$.
14. $m^2 - 3m - 28$.
15. $m^2 + 3m - 28$.
16. $y^2 + 6y - 40$.
17. $y^2 - 6y - 40$.
18. $t^2 - \frac{3}{2}t + \frac{1}{2}$.
19. $x^2 + 6x + 5$.
20. $x^2 + 9x + 20$.
21. $x^2 - 8x + 15$.
22. $x^2 + 8x + 7$.
23. $x^2 - 10x + 9$.
24. $x^2 + 7x + 12$.
25. $x^2 - 5x - 14$.
26. $x^2 + 2x - 15$.
27. $s^2 - \frac{3}{4}s - \frac{1}{4}$.
28. $(a+b)^2 - 10(a+b) + 9$.
29. $(m+n)^2 + 2(m+n) - 15$.

VIII. TYPE $mx^2 + px + q$

164. The type $mx^2 + px + q$ may be factored by a comparison of factors similar to that in Section 163. This trial method is commonly called *the method of cross products.*

EXAMPLE

Factor: $12x^2 - 7x - 10.$

From the factors of the first and third terms without taking into account the middle term, the possible factors are

$$\begin{array}{c} 12x - 10 \\ \underline{\quad x + 1} \end{array} \qquad \begin{array}{c} 2x - 5 \\ \underline{6x + 2} \end{array} \qquad \begin{array}{c} 3x + 2 \\ \underline{4x - 5} \end{array}$$

in which the two coefficients in any column may be interchanged.

To find which are the actual factors, it is only necessary to multiply the coefficients and observe what combination produces the coefficient of the second term of the original trinomial, in this case -7.

Thus, $$\begin{array}{c} 12 - 10 \\ \times \\ 1 + 1 \\ \hline -2 \text{ (not } -7) \end{array} \qquad \begin{array}{c} 2 - 5 \\ \times \\ 6 + 2 \\ \hline -26 \text{ (not } -7) \end{array} \qquad \begin{array}{c} 3 + 2 \\ \times \\ 4 - 5 \\ \hline -7 \text{ (correct)} \end{array}$$

Therefore, the factors are $3x + 2$ and $4x - 5$. Test as usual.

It is seldom necessary to try all the sets of factors in their different combinations. Simple conditions will eliminate them at once. For example, $12x - 10$ could not be a factor, because, if it were, the factor 2 which it contains would be a factor of the given trinomial, and it is not. For the same reason, $6x + 2$, in the second set, could not be a factor.

WRITTEN EXERCISES

Factor:

1. $6x^2 - 25x + 24.$
2. $4x^2 - 27x - 7.$
3. $12x^2 + 11x - 5.$
4. $3p^2 + 16p - 35.$
5. $88p^2 - 31p - 15.$
6. $24x^2 + 73x + 24.$
7. $7y^2 + 48y - 7.$
8. $10y^2 + 13y - 3.$
9. $21x^2 + 46x - 7.$
10. $6x^2 + 47x + 35.$
11. $8p^2 - 30p + 7.$
12. $20y^2 - 49y + 30.$
13. $18x^2 - 19x + 5.$
14. $24z^2 - 103z + 55.$

15. $3m^2 + 20m + 32$.
16. $21y^2 - 61y + 28$.
17. $26p^2x^2 - 10px - 36$.
18. $15m^2 + 8m - 23$.
19. $15z^2 - 17z - 42$.
20. $22x^2 - 19x - 21$.
21. $26p^2x^2 + 62px + 36$.
22. $3p^2 - 20p + 32$.
23. $15a^2 + 17a - 42$.
24. $25x^2 - 5x - 56$.
25. $15m^2 - 38m + 23$.
26. $21b^2 - 37b - 28$.
27. $25c^2 + 75c + 56$.
28. $22x^2 + 19x - 21$.
29. $25z^2 + 5z - 56$.
30. $21y^2 + 37y - 28$.
31. $3m^2 - 4m - 32$.
32. $22x^2 + 47x + 21$.

165. The methods for factoring given in Sections 162, 163, and 164 are of limited value, for they determine the factors only in favorable instances. Thus, $x^2 - 17x + 72$ was readily factored in Section 163, but $x^2 - 16x + 72$ could not be so factored. A general method will be given in Chapter XXX by means of which all such trinomials can readily be factored.

REVIEW EXERCISES

Factor:

1. $x^8 - 49y^4$.
2. $200 - 2x^4$.
3. $a^{12}b^9 - bc^{16}$.
4. $243 - 3x^{12}$.
5. $121a^{2p} - y^2$.
6. $a^8 - 256b^8$.
7. $1 - 25(a + 2b)^2$.
8. $1 - 81(x^2 + x - 1)^2$.
9. $2a^2 - 4a - 2a^4 + 2$.
10. $14x^7 - 21x^5y$.
11. $3a^nb^3 - 6a^{2n}b^4 + 15a^{3n}b^5$.
12. $x^5 - 2x^4 + x^3$.
13. $b^3z - 2b^2z^2 + bz^3$.
14. $4a^3 - 48a^2b^2 + 144ab^4$.
15. $x^my^5z^{2p} + 7x^my^3z^{2p-1}$.
16. $100a^4 - 61a^2 + 9$.
17. $a^2 - (b + 1)^2$.
18. $z^5 + z - 2z^3$.
19. $25(a - b)^2 + 20c(a - b) + 4c^2$.
20. $a^2 + b^2 + c^2 - 2ab - 2bc + 2ac$.
21. $a^2 + b^2 - x^2 - y^2 + 2ab - 2xy$.
22. $2ab - b^2 + x^2 - a^2$.
23. $16(2x - y)^2 - (x - 3y)^2$.
24. $4a^2 + 9b^2 + 25c^2 - 12ab + 20ac - 30bc$.

25. $a^4 - 169\,b^{10}$.

26. $36\,c^4 + 47\,c^2d^2 + 16\,d^4$.

27. $a^5 - 20\,a + a^3$.

28. $x^{2p} - x^p - 72$.

29. $ay^2 - ay - 20\,a$.

30. $2\,x^3 - 10\,x^2 + 12\,x$.

31. $x^2y^2 - 3\,xy - 88$.

32. $(x^2 - 1)^2 - 2\,(x^2 - 1) - 63$.

33. $8\,a^2 + 2\,a - 15$.

34. $25\,x^4 - 41\,x^2y^2 + 16\,y^4$.

35. $9\,z^4 - 148\,z^2 + 64$.

36. $14\,x^2 - 39\,x + 10$.

37. $7\,x^2 - 3\,x - 4$.

38. $10 - 5\,a - 15\,a^2$.

39. $3\,t^2 + 7\,t - 20$.

40. $10\,s^2 - 34\,s + 28$.

41. $6\,c^2 - 5\,cd - 21\,d^2$.

42. $m^2 + 26\,m + 133$.

43. $(a - b)^2 - 15\,(a - b) + 50$.

44. $21\,r^2 - 18\,rs - 48\,s^2$.

45. $x^4y^4 + 2\,x^2y^2 - 63$.

46. $9\,x^{2k} - x^ky - 10\,y^2$.

47. $4\,x^4 - 9\,x^2 + 6\,x - 1$.

48. $(2\,a + b)^2 - 16(3\,a - b)^2$.

49. $(3 + b)^2 - 2\,(3 + b)(x - 1) + (x - 1)^2$.

50. $4\,x^{4n+2}y^{3n+4} - 8\,x^{5n+4}y^{3n+3} + 12\,x^{5n+3}y^{5n+6}$.

IX. TYPE x^3+y^3 AND x^3-y^3

166. We know by multiplying that

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3,$$

and that

$$(x - y)(x^2 + xy + y^2) = x^3 - y^3.$$

Hence, one factor of the sum of two cubes is the sum of the numbers, and the other is the sum of the squares of the numbers minus their product.

Also one factor of the difference of two cubes is the difference between the numbers, and the other is the sum of their squares plus their product.

EXAMPLES

1. **Factor:** $27\,x^3 + 8\,y^3$.

$$27\,x^3 = (3\,x)^3.$$
$$8\,y^3 = (2\,y)^3.$$
$$\begin{aligned}\therefore\ 27\,x^3 + 8\,y^3 &= (3\,x + 2\,y)[(3\,x)^2 - 3\,x \cdot 2\,y + (2\,y)^2]\\ &= (3\,x + 2\,y)(9\,x^2 - 6\,xy + 4\,y^2).\end{aligned}$$

TEST by multiplication.

2. Factor: $8\,a^3b^3 - 125\,c^3.$

$$\begin{aligned} 8\,a^3b^3 &= (2\,ab)^3.\\ 125\,c^3 &= (5\,c)^3.\\ \therefore\ 8\,a^3b^3 - 125\,c^3 &= (2\,ab - 5\,c)[(2\,ab)^2 + 2\,ab \cdot 5\,c + (5\,c)^2]\\ &= (2\,ab - 5\,c)(4\,a^2b^2 + 10\,abc + 25\,c^2). \end{aligned}$$

Test by letting $a = b = c = 1$: $8 - 125 = -3 \cdot 39$.

3. Factor: $8\,x^6 - a^6.$

$$\begin{aligned} 8\,x^6 - a^6 &= (2\,x^2)^3 - (a^2)^3\\ &= (2\,x^2 - a^2)[(2\,x^2)^2 + (2\,x^2)(a^2) + (a^2)^2]\\ &= (2\,x^2 - a^2)(4\,x^4 + 2\,a^2x^2 + a^4). \end{aligned}$$

167. Types $x^3 - y^3$ and $x^3 + y^3$ may be used in calculation.

EXAMPLE

Calculate: $14^3 - 13^3.$

$$\begin{aligned} 14^3 - 13^3 &= (14 - 13)(14^2 + 14 \cdot 13 + 13^2)\\ &= 14^2 + 14 \cdot 13 + 13^2\\ &= 14(14 + 13) + 13^2\\ &= 14 \cdot 27 + 13^2\\ &= 378 + 169\\ &= 547. \end{aligned}$$

WRITTEN EXERCISES

Factor:

1. $a^3 - b^3.$
2. $a^3 - 8\,b^3.$
3. $8\,a^3 + b^3.$
4. $27\,x^3 - y^3.$
5. $y^3 + 27\,z^3.$
6. $a^{3n} + 1.$
7. $\dfrac{1}{a^3} - 1.$
8. $a^3 - 1.$
9. $x^6 - y^6.$
10. $a^6 + b^6.$
11. $64\,a^6 - 1.$
12. $8\,x^3 + 1.$
13. $27\,y^{6p} - 1.$
14. $a^3 - \frac{1}{27}.$
15. $xy^4 - x^4y.$
16. $125\,a^6 - b^3.$
17. $(a + b)^3 - 1.$
18. $27\,x^3y^3z^3 + 8.$
19. $a^3b^3 + (a + b)^3.$
20. $8\,m^3 - 27\,p^3q^3.$
21. $10^3 - h^{3n}.$
22. $(a + b)^3 - (b - c)^3.$
23. $(3\,a + b)^3 - (2\,a - b)^3.$
24. $(x + y)^3 - (x - y)^3.$
25. $(c + d)^3 + (2\,c - d)^3.$
26. $(x^3 + 1)^3 - (y^3 + 1)^3.$
27. $8\,a^3 + 125(b + c)^3.$

Calculate:

28. 16^3-15^3. **29.** 19^3-18^3. **30.** 23^3-21^3. **31.** 57^3-56^3.

32. It is known that the volume of a sphere is $\frac{4}{3}\pi r^3$, r being the length of the radius. Using $\frac{22}{7}$ as an approximate value of π, calculate the number of cubic inches in a spherical shell whose outer radius is 14 in., and inner radius 13 in.

Solution. The volume of the outer sphere is $\frac{4}{3}\cdot\frac{22}{7}\cdot 14^3$, and that of the inner sphere is $\frac{4}{3}\cdot\frac{22}{7}\cdot 13^3$.

Hence the volume of the shell in cubic inches is

$$\tfrac{4}{3}\cdot\tfrac{22}{7}(14^3-13^3)=\tfrac{4}{3}\cdot\tfrac{22}{7}\cdot 547=2292.^{+}$$

33. Find similarly the volumes of spherical shells if:

	(1)	(2)	(3)	(4)
Outer radius =	16	19	25	36
Inner radius =	15	18	23	32

168. We have seen how to factor the difference of two squares and the difference of two cubes; it will be sufficient for present purposes to note that:

The difference of any two even powers is always divisible by the difference of their square roots, and the difference of two odd powers is divisible by the difference of the numbers.

Furthermore, the sum of any two odd powers is divisible by the sum of the numbers.

EXAMPLES

1. Factor: $x^{10}-y^{10}$.

$x^{10}-y^{10}=(x^5)^2-(y^5)^2=(x^5-y^5)(x^5+y^5)$. Then x^5-y^5 is divisible by $x-y$, and x^5+y^5 is divisible by $x+y$.

2. Factor: x^5-y^5.

$$x^5-y^5=(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$$

3. Factor: a^5+32b^5.

$$a^5+32b^5=(a+2b)(a^4-2a^3b+4a^2b^2-8ab^3+16b^4)$$

WRITTEN EXERCISES

Factor into two binomials:

1. $m^2n^2 - p^2q^2$.
2. $4\,m^2 - 9\,n^2$.
3. $16 - y^2$.
4. $4\,a^4b^2 - 1$.
5. $m^4 - p^4$.
6. $x^4y^4 - 64$

Factor into three binomials:

7. $a^4 - b^4$.
8. $16 - p^4$.
9. $x^4y^4 - 1$.
10. $81\,m^4 - p^4$.

State one factor of each of these expressions and prove by division that it is a factor:

11. $m^5n^5 - 1$.
12. $x^7 - a^7$.
13. $x^{3a} + y^{3a}$.
14. $x^5 - 32$.
15. $8\,a^3b^3 + 27$.
16. $x^{5a} + 1$.

REVIEW

169. In attempting to factor an expression not an indicated root:

First. Remove any monomial factor, as in Type II.

Second. If the resulting expression is a binomial, apply Type V, or VI, or IX.

Third. If the resulting expression is a trinomial, apply Type IV, or VII, or VIII.

Fourth. If the expression is not one of the above types, attempt to group it, as in Type III.

WRITTEN EXERCISES

1. $3\,a^2 - 15\,ab$.
2. $a^2 - x^2$.
3. $p^2 - 9\,q^2$.
4. $3\,a^4 - 6\,a^2b$.
5. $xy^2z - xyz^2$.
6. $16\,a^4 - 9\,c^4$.
7. $a^3 - x^3$.
8. $x^3 + z^3$.
9. $a^2x^4 - 2\,ax^2 + 1$.
10. $m^2 + 6\,m + 5$.
11. $x^6 + 1$.
12. $x^2 - 2\,ax - 3\,a^2$.
13. $y^{3b} + 1$.
14. $ac - ad + bc - bd$.

15. $2x^3 - 54$.
16. $-x^4 + x^2 + 12$.
17. $2x^2 + 5bx - 12b^2$.
18. $3a^2 + 12ab - 2a - 8b$.
19. $x^4 - y^4$.
20. $4a^2 - b^2 + 6a - 3b$.
21. $a^2 - 4b^2 - ac + 2bc$.
22. $25x^4 - 10x^2y + y^2 - 9z^2$.
23. $x^4 - 6x^2y^2 + y^4$.
24. $16a^4 + 24a^2b + 9b^2$.
25. $4x^2 - 4xy + y^2 - x^2y^2$.
26. $25a^6 + 20a^3 + 4$.
27. $a^9 + b^9$.
28. $3x^2 - 7xy + 2y^2$.
29. $3x^2 + 6xy - 24y^2$.
30. $x^8 + 3x^4 - 4$.
31. $3x^3 - 6x^2 - 9x$.
32. $3a^2 + 6ab - 24b^2$.
33. $x^{2m} - 2x^m - 3$.
34. $a^2 - 2ab - ac + 2bc$.
35. $m^4 - 11m^2n^2 + n^4$.
36. $a^5 + b^5$.
37. $x^4 - 3x^2y^2 + y^4$.
38. $8 + z^6$.
39. $4x^4 + 1$.
40. $x^8 - y^8$.
41. $a^2 - b^2 - (a - b)^2$.
42. $27x^2 + 3x - 2$.
43. $x^{2n} - y^{2b}$.
44. $a(a + b) - c(c + b)$.
45. $ab - 3a - 2b + 6$.
46. $3x^3 - 6x^2 - 9x$.
47. $2a^3 - 3a^2 - 2a + 3$
48. $4x^4 + y^4 - 5x^2y^2$.
49. $a^4 - 3a^2 + 1$.
50. $4x^2 + 11x - 20$.
51. $x^2 - 4ax - 4b^2 + 8ab$.
52. $a^3 + a^2 - a - 1$.
53. $x^{2a} + 2x^a + 1$.
54. $(a + b)^2 - (c - d)^2$.
55. $b^2 - a^2 + 2ac - c^2$.
56. $a^2 - b^2 - a - b$.
57. $\frac{c^2}{d^2} + \frac{2c}{d} - 3$.
58. $16x^2 + 10xy - 9y^2$.
59. $x^8 + x^4 + \frac{1}{4}$.
60. $x^6 - x^4 - x^2 + 1$.
61. $x^3 - 216$.
62. $a^4 + a^2b^2 + b^4$.
63. $x^3 + 8$.
64. $2a^2 + 13a - 24$.
65. $a^{2n} + 6a^n + 9$.
66. $x^4 - 81$.
67. $6x^2 - 11xy - 2y^2$.
68. $6x^2 - 13xy + 6y^2$.
69. $xy^7 - xy$.
70. $x^{2p} - 2x^py + y^2$.
71. $x^4 + x^2y^4 + y^8$.
72. $x^4 - 2x^2 + 1$.
73. $a^2b - a^2 - ab + a$.
74. $a^4 - 6a^2 + 1$.
75. $x^8 + x^4y^2 + y^4$.
76. $x^{2a} - 4x^a + 4$.

77. $\frac{a^2}{4} - ab + b^2$.

78. $1 + a - b - ab$.

79. $12\,x^2 - 27\,y^2$.

80. $64 - a^6$.

81. $3\,a^2 - 15\,a + 18$.

82. $a^2 + a - 30$.

83. $x^2 + 3\,x + 2$.

84. $16 + 4\,a^2 + a^4$.

85. $x^{3p} + y^{3q}$.

86. $16\,x^2 - 48\,x + 35$.

87. $a^4 + a^2 + 1$.

88. $2\,a^2 + 3\,ab - 2\,b^2$.

89. $2\,x^2 + xy - 3\,y^2$.

90. $6\,a^2 + 10\,ab - 4\,b^2$.

91. $a^{x+1} - a^{x-2}b^3$.

92. $p^3 + 3\,p^2 + 3\,p + 1$.

93. $x^4 + 2\,x^3 + 2\,x^2 + 2\,x + 1$.

94. $\frac{1}{x^2} + 1 + x^2$.

95. $2\,x^3 + 3\,x^2 - 2\,x - 3$.

96. $4\,b^3 - 9\,b$.

97. $55\,x^2 - x - 2$.

98. $6\,y^3 - y^2 + 6\,y - 1$.

99. $\frac{1}{9}\,x^2 - \frac{4}{25}\,y^2$.

100. $a^4 - 3\,a^2 + 1$.

101. $x^3 - x^2 - x + 1$.

102. $a^2 + \frac{5}{6}\,a + \frac{1}{6}$.

103. $1 - \frac{1}{a^2}$.

104. $2\,a^2 + ab - 6\,b^2$.

105. $12\,a^2 - 5\,ab - 3\,b^2$.

106. $x^{2n} - 2\,x^n + 1$.

107. $abx^2 + (a^2 + b^2)\,xy + aby^2$.

108. $a^3x - a^2c + a^2by - ab^2x - b^3y + cb^2$.

109. $(a + b)(c^2 - d^2) - (a^2 - b^2)(c - d)$.

110. $a^2 + b^2 + c^2 + 2\,bc + 2\,ca + 2\,ab$.

111. $a^2 + b^2 + 1 + 2\,b + 2\,a + 2\,ab - d^2$.

Factor by reference to:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$$

112. $m^2 + q^2 + r^2 + 2\,mq + 2\,mr + 2\,qr$.

113. $a^2 + b^2 + x^2 + 2\,ab - 2\,ax - 2\,bx$.

114. $x^2 + y^2 + 25 + 2\,xy + 10\,x + 10\,y$.

115. $9 + a^2 + m^2 + 2\,am - 6\,a - 6\,m$.

116. $t^2 + 2\,tv + v^2 - 6\,t + 9 - 6\,v$.

117. $25\,x^2 + 9\,y^2 + 4\,z^2 + 30\,xy - 20\,xz - 12\,yz$.

118. $x^4 - 2\,x^2a^2 - 2\,x^2b^2 + a^4 + b^4 + 2\,a^2b^2$.

CHAPTER XIII

EQUATIONS

170. Factoring is an important process in the solution of equations.

171. PREPARATORY.

1. Find the values of the trinomial x^2-x-2 when $x=1$; 2; 0; -1; -2; 5; for which values of x does it become 0?

2. Find the value of the binomial x^2-2x when $x=1$; -1; 2; -2; 0; 3; for which values of x does it become 0?

3. According to Exercise 1, what are the roots of the equation $x^2-x-2=0$?

4. According to Exercise 2, what are the roots of the equation $x^2-2x=0$?

5. What roots are common to the two equations?

172. Equivalent Equations. If two equations have the same roots, the equations are said to be **equivalent.**

Thus, $$4x=12$$
and $$5x-15=0$$
are equivalent, each having the root 3, and no others.

Also, $$x^2-25=0$$
and $$4x^2-100=0$$
are equivalent, each having the root 5, -5, and no others.

But, $$x^2-25=0$$
and $$x^2-8x+15=0$$
are not equivalent, for the first has the roots 5, -5, while the second has the roots 5 and 3.

WRITTEN EXERCISES

Write the roots of these equations and find which exercises contain equivalent equations:

1. $3x - 6 = 0$, $x - 3 = 4$.
2. $x - 3 = 5$, $2x = 16$.
3. $x^2 = 4$, $x + 2 = 0$.
4. $x^2 = 2$, $2x^2 = 4$.
5. $x - 4 = 8$, $x + 2 = 6$.
6. $x^2 = 9$, $x + 3 = 0$.

173. If two equations have between them all the roots of a third equation and no others, the two together are said to be equivalent to the third.

EXAMPLES

1. One of the equations $x - 5 = 0$, and $x - 3 = 0$, has the root 5, the other the root 3. Between them, they have the roots 3 and 5, which are the roots of $x^2 - 8x + 15 = 0$. The equations $x - 3 = 0$ and $x - 5 = 0$ are together equivalent to $x^2 - 8x + 15 = 0$.

2. The equation $(x - 1)(x - 2) = 0$ asks: *For what values of x does the product* $(x - 1)(x - 2)$ *have the value zero ?*

The product is zero, if either factor is zero, and not otherwise (Sec. 114). $\therefore (x - 1)(x - 2) = 0$, if $x - 1 = 0$, or if $x - 2 = 0$, and not otherwise.

Thus, the solution of the equation $(x - 1)(x - 2) = 0$ depends upon the solution of $x - 1 = 0$ and $x - 2 = 0$. The roots of these being 1 and 2, the roots of $(x - 1)(x - 2) = 0$, are likewise 1 and 2.

The pair of equations $x - 1 = 0$, $x - 2 = 0$ is equivalent to the equation $(x - 1)(x - 2) = 0$.

ORAL EXERCISES

State the equations of the first degree that are equivalent of each of the following:

1. $(x - 3)(x - 2) = 0$.
2. $(x - 5)(x - 3) = 0$.
3. $(x - 3)(x + 2) = 0$.
4. $(x + 5)(x + 3) = 0$.
5. $(x - 3)(x + 10) = 0$.
6. $x(x - 5) = 0$.
7. $(x + 7)(x + 1) = 0$.
8. $x(x + 3) = 0$.
9. $x(x - a) = 0$.
10. $(x + 8)(x - 11) = 0$.

Factor the first member of each of the following equations and state the equations equivalent to each:

11. $x^2 - 3x + 2 = 0.$

12. $x^2 - 5x + 4 = 0.$

13. $x^2 - 6x + 5 = 0.$

14. $x^2 + 6x + 5 = 0.$

15. $x^2 + 8x + 7 = 0.$

16. $x^2 - 4x - 5 = 0.$

17. $x^2 - 9 = 0.$

18. $x^2 - 2x = 0.$

19. $3x^2 - 2x = 0.$

20. $x^2 - 14x + 33 = 0.$

WRITTEN EXERCISES

1–10. Solve the equation in each exercise above from 11–20 by solving the equivalent equations.

Find the roots of each equation by factoring the left member and solving the equivalent equations:

11. $x^2 - x - 20 = 0.$

12. $x^2 + x - 30 = 0.$

13. $x^2 - 3x - 18 = 0.$

14. $x^2 - x - 30 = 0.$

15. $x^2 + 3x - 18 = 0.$

16. $x^2 - x - 42 = 0.$

17. $x^2 - 7x - 18 = 0.$

18. $x^2 - 17x + 72 = 0.$

19. $x^2 - x - 56 = 0.$

20. $x^2 + 7x - 18 = 0.$

21. $x^2 + 11x + 28 = 0.$

22. $x^2 - 3x - 28 = 0.$

23. $x^2 + 6x - 40 = 0.$

24. $x^2 + 11x + 24 = 0.$

25. $x^2 - 3x - 28 = 0.$

26. $x^2 - 6x - 40 = 0.$

27. $x^2 + 18x + 81 = 0.$

28. $x^2 - 20x + 100 = 0.$

29. $x^2 - 8x + 15 = 0.$

30. $x^2 + 9x + 20 = 0.$

31. $x^2 + 8x + 7 = 0.$

32. $x^2 + 2x - 15 = 0.$

33. $x^2 - x - 6 = 0.$

34. $x^2 - 5x - 14 = 0.$

35. $x^2 + x - 110 = 0.$

36. $x^2 - 5x - 24 = 0.$

37. $-x^2 + x + 12 = 0.$

38. $3x^2 - 7x + 2 = 0.$

39. $4z^2 + 12z + 9 = 0.$

40. $25x^2 + 20x + 4 = 0.$

41. $x^2 - x - 2 = 0.$

42. $x^2 - 64 = 0.$

43. $1 - x^2 = 0.$

44. $2y^2 + 12y + 10 = 0.$

174. Quadratic Equations. Equations of the second degree are called **quadratic equations.**

For example, $x^2 = 16$, $x^2 - 3x = 0$, and $x^2 - 5x + 6 = 0$ are quadratic equations with one unknown, x.

175. The way in which quadratic equations occur in problems is illustrated in the following exercises.

WRITTEN EXERCISES

Translate each statement into an equation:

1. The product of a certain number and the number increased by 3 is 70.

2. The product of two consecutive integers is 132.

3. The area of a rectangle whose length is three times its height is 75 sq. in.

4. In the case of a body falling from rest, the distance d fallen in the time t is one half the product of a fixed number g (the constant of gravity) and the square of the time.

176. General Form. The *general form* of the quadratic equation is

$$ax^2 + bx + c = 0,$$

in which a, b, c, are any known numbers, except that a may not be zero.

For example.

	a	b	c
$3x^2 - x + 5 = 0,$	3	-1	5
$x^2 + 7x = 0,$	1	7	0
$4x^2 - 12 = 0,$	4	0	-12
$x^2 = 0,$	1	0	0

177. It is often necessary to simplify equations apparently involving x^2 to see whether or not a is zero; that is, whether or not the equations are really quadratics.

For example:

$\frac{x^2 - 3}{5x} = \frac{x + 2}{7}$ can more readily be seen to be a quadratic equation when reduced to $2x^2 - 10x - 21 = 0$.

In this form it is apparent that $a = 2$, $b = -10$, $c = -21$.

WRITTEN EXERCISES

Reduce each equation to the type form and write the value of a; of b; of c:

1. $\frac{x^2-1}{2}=1.$
2. $3x^2-5=\frac{x-1}{2}.$
3. $\frac{x^2-1}{2}=\frac{x^2-4}{5}.$
4. $\frac{2x^2+3}{5}=\frac{x^2}{7}.$
5. $x+\frac{1}{x}=3.$
6. $12-x=\frac{x^2}{8}.$
7. $\frac{x+6}{7}=\frac{x^2}{2}.$
8. $\frac{2x-9}{3}=\frac{2x^2}{5}.$

178. Incomplete Quadratic Equations. A quadratic equation which lacks either its absolute term or its term in x, that is, in which either c or b is zero, is an **incomplete quadratic equation.** The two forms of the incomplete quadratic equation are

$$x^2+\frac{c}{a}=0.$$

$$ax^2+bx=0.$$

It is unnecessary to consider the case where b and c are both 0, because x would always be 0 whatever the value of a.

179. Solution of Incomplete Quadratic Equations.

I. *The incomplete quadratic equation* $x^2+\frac{c}{a}=0$ *is solved by transposing and extracting the square root of both members.*

EXAMPLES

1. $x^2=4.\quad \therefore\ x=\sqrt{4}=\pm 2.$
2. $3x^2=75.\quad \therefore\ x^2=25$ and $x=\pm 5.$
3. $ax^2=b.\quad \therefore\ x^2=\frac{b}{a}$ and $x=\pm\sqrt{\frac{b}{a}}.$

In Example 3 the value of x is the square root of the fraction $\frac{b}{a}$, and it is sufficient to indicate this by the use of the radical sign.

II. *The incomplete quadratic equation* $ax^2 + bx = 0$ *is solved by factoring* (Sec. 173).

Thus, $x(ax + b) = 0$, in which $x = 0$ and $ax + b = 0$ from which $x = -\frac{b}{a}$

One value of x in this case is always zero (Sec. 114), and the other is the root of the linear equation $ax + b = 0$.

EXAMPLES

1. $x^2 - x = 0.\quad \therefore x(x-1) = 0$ and $x = 0$, $x = 1$.
2. $3x^2 - 10x = 0.\quad \therefore x(3x - 10) = 0$ and $x = 0$, $x = \frac{10}{3}$.

WRITTEN EXERCISES

Solve the following equations:

1. $x^2 = 169$.
2. $x^2 - 121 = 0$.
3. $x^2 - 144 = 0$.
4. $x^2 - 81 = 0$.
5. $x^2 - 49 = 0$.
6. $x^2 - 625 = 0$.
7. $3x^2 - 75 = 0$.
8. $4x^2 - 100 = 0$.
9. $5x^2 - 500 = 0$.
10. $12x^2 - 1728 = 0$.
11. $s^2 - \frac{1}{4} = 0$.
12. $t^2 - \frac{9}{16} = 0$.
13. $r^2 - 25 = 0$.
14. $121x^2 = 1089$.
15. $7x^2 - 448 = 0$.
16. $w^2 - \frac{16}{25} = 0$.
17. $2s^2 - \frac{8}{9} = 0$.
18. $3x^2 - \frac{12}{25} = 0$.
19. $3x^2 + 8 = 5x^2 + 8$.
20. $17x^2 + x = 0$.
21. $40x^2 - 25x = 0$.
22. $9x^2 - 17 = 4x^2 - 12$.
23. $29x^2 - 30 = 10x^2 + 46$.
24. $40x^2 - 43 = 7 - 10x^2$.
25. $7x^2 - 5 = 4x^2 + 7$.
26. $3(x^2 - x) = 2x^2 - 3x + 1$.

27. In the equation $16t^2 = 256$, t is the number of seconds required for a body to fall 256 ft. How many seconds is this?

28. The area inclosed by a circle is πr^2. Using 3.1416 as the approximate value of π, find the radius of a circle whose area is 12.5664 sq. in.

29. Find the radius of a circle whose area is 3.1416 sq. ft.

30. Find the diameter of a circle whose area is 28.2744 sq. yd.

31. If a stands for the area of a circle of radius r and π be taken as $\frac{22}{7}$, $a = \frac{22}{7}r^2$. The area of a certain circle is $\frac{22}{7}$ sq. ft. Find the length of the radius in feet.

32. Find the radius of a circle whose area is $\frac{22}{63}$ sq. in.; $\frac{22}{175}$ sq. in.; $\frac{22}{112}$ sq. yd.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What are *equivalent equations?* Sec. 172.
2. What is a *quadratic equation?* Sec. 174.
3. State the *general form* of quadratic equations. Sec. 176.
4. What is an *incomplete quadratic equation?* Illustrate its two forms. Sec. 178.
5. How may each kind of incomplete quadratic equations be solved? Sec. 179.

HISTORICAL NOTE

We have seen that the solution of equations and the factoring of algebraic expressions are closely related, and we have found it advantageous to employ factoring in solving equations. If Diophantos had known this method, he would have found two roots of the quadratic equation instead of stopping with one. For example, he showed in his work *Arithmetica* that $(x-1)(x-2) = x^2 - 3x + 2$, and could find one root of equations like $x^2 - 3x + 2 = 0$, but he did not discover that this equation is the same as $(x-1)(x-2) = 0$, which evidently has two roots, 1 and 2.

It is strange that the application of factoring to solving equations was so long overlooked, for it remained unknown long after other more obscure processes were discovered. To Thomas Harriot, an English algebraist of the seventeenth century, belongs the credit of first reducing an equation by factoring. The English were proud to boast of Harriot's knowledge of mathematics, and Sir Walter Raleigh sent him to Virginia to survey the new colonial territory. He afterward returned to England and made other improvements in algebra, among them the use of small letters in place of capital letters to represent numbers; but Harriot's ignorance of negative number prevented him from applying the factoring process to any but particular cases of equations.

CHAPTER XIV

FACTORS AND MULTIPLES

180. Integral Expressions. Algebraic expressions which do not contain fractions are called integral expressions, but this classification is generally understood to refer to the letters involved.

Thus, a, $2ab$, a^2b, $a+x$, $(2x-y)^2$, are integral expressions.

And, $\frac{1}{2}a$, $\frac{1}{a}$, $\frac{b}{3a}$, $\frac{2a-b}{c}$, $\frac{c-3b}{4}$, are fractional expressions, although $\frac{1}{2}a$ and $\frac{c-3b}{4}$ are integral with respect to the letters involved.

Equations are called "integral" whenever the fractions involved are confined to the coefficients.

ORAL EXERCISES

Select the fractional expressions from the following:

1. $a+\frac{1}{2}b.$
2. $x+\frac{1}{b}.$
3. $3x+\frac{2}{c}.$
4. $\frac{a+b+c}{5}.$
5. $\frac{1}{a}(b+c).$
6. $\frac{x-y+z}{2a}.$
7. $\frac{2y}{3}+q.$
8. $\frac{4x+8y}{4}.$
9. $\frac{2a}{2c}-\frac{3b}{5d}.$
10. $a-\frac{1}{3}x+\frac{4}{3}y.$

11. In Exercises 1–10, select the expressions that are fractional with respect to the coefficients only.

12. From the same Exercises, select the expressions that are fractional with respect to the letters only.

13. Select all the fractional expressions after all possible reductions have been made.

181. Common Factor. An expression that is a factor of each of two or more expressions is called a common factor of the expressions.

182. This chapter is concerned with integral factors only.

183. Algebraic expressions which have no common literal factors are said to be algebraically prime to each other.

184. PREPARATORY.

1. What is the degree of $2a^2$? Of $3a^2b$? Of $(a+x)^2$? (See Sec. 33, 34.)

2. What is the degree of $3a^2b$ with respect to a? With respect to b? With respect to ab?

185. Highest Common Factor. The highest common factor (h. c. f.) of two or more algebraic expressions is the algebraic expression of highest degree that is an exact divisor of each expression, including both the numerical and literal parts.

For example, to find the h. c. f. of $3a^2b$, $-6ab^2$, and $9abc$:

The literal common factor of highest degree is ab. The greatest common divisor (g. c. d.) of 3, -6, and 9 is 3. Hence the h. c. f. of $3a^2b$, $-6ab^2$, and $9abc$ is $3ab$.

Although -3 is also a common divisor of 3, -6, and 9, it is customary to take the g. c. d. with the positive sign.

If the given expressions are factored so as to have the h. c. f. as one factor, the set of second factors will have no further common factor, other than unity.

186. *In the case of monomials,* the h. c. f. is seen by inspection. Its coefficient is the g. c. d. of the given numerical coefficients, and its literal part is the product of all the different letters, each with the lowest exponent that it has in any of the monomials.

187. *If expressions not monomials* are given, they must first be factored if possible, after which the h. c. f. can usually be seen.

1. Find the h. c. f. of $ab^2 + abc$, and $b^2c + bc^2$:

Factoring,
$$ab^2 + abc = ab(b + c).$$
$$b^2c + bc^2 = bc(b + c).$$

The h. c. f. is the product of the common factors b and $b+c$, or $b(b+c)$.

2. Find the h. c. f. of $(1-x)^2$, x^2-1, x^2-2x+1.

Factoring,

$$\begin{aligned}(1-x)^2 &= (1-x)(1-x).\\ x^2-1 &= (x-1)(x+1).\\ x^2-2x+1 &= (x-1)(x-1).\end{aligned}$$

The h. c. f. is $1-x$ or $x-1$, because these factors differ only in sign and the h. c. f. may have either sign.

188. The h. c. f. of expressions not readily factored can be found by the long division process, but no problems will be given in this chapter requiring the general method.

189. To find the h. c. f.: *Factor each expression into its prime factors. Then find the product of all the common prime factors, using each the least number of times it occurs in any of the given expressions.*

WRITTEN EXERCISES

Find the h. c. f. of:

1. $3xy^3$, x^2y.
2. x^4y^3, x^3y, xy^3.
3. $10a^2$, $15a^3$, $5a$.
4. x^2+xy, $(x+y)^3$.
5. $3a^2+3ab$, a^2-b^2.
6. ax^4-ay^4, x^2-y^2.
7. $3x^4$, $2x^3$, $4x^5$, x^3.
8. $3a^2bc^3$, $15ab^3c^2$, $10a^2b^2$.
9. $4a^3b^2c$, $8ab^5c^2$, $12axybc^2$
10. $10a^2b^4c$, $15ab^3c^2$, $20a^2c^3$.
11. a^4+4, a^2-2a+2.
12. a^3-1, a^2+a+1.
13. $3ax^2$, $-2a^2x$, a^2x^2, $-3abx$.
14. $3a^3+2a^2b-5ab^2$, $2a^2b+2ab^2$.
15. x^2-a^2, $x^2-2ax+a^2$.
16. $a^2x^2-8ax+16$, $5ax-20$.
17. $ax+ay+bx+by$, $x^2+2xy+y^2$.
18. $a^2-b^2x^2$, $a^2-2abx+b^2x^2$.
19. x^2+2x+1, x^3+x^2+x+1.
20. $6x^2+19x+10$, $3x^2-13x-10$.
21. $(a+b)^3-1$, $a+(b-1)$.
22. $8x^3-1$, $4x^2-4x+1$.
23. $x^2-8x+15$, x^2-5x.

24. $x^2+8x+15,\ x^2+3x.$
25. $3x^2+x-2,\ 3x^2-2x.$
26. $a^3-x^3,\ a^3-3a^2x+3ax^2-x^3.$
27. $x^2+5x+6,\ x^2-4.$
28. $x^2-2xy-3y^2,\ 2x-6y.$
29. $a^2-a-6,\ a^3-9a^2+27a-27.$
30. $a^2-16,\ 16-8a+a^2.$
31. $m^2-2mr+r^2,\ m^3-r^3.$
32. $a^2+2ap+p^2,\ ab+bp.$
33. $d^2-9,\ d^2+6d+9.$
34. $(a-b)^3,\ a^2-b^2.$ 35. $a^3+x^3,\ (a+x)^2.$ 36. $a^2-ay,\ a^4-y^4.$

190. Multiples. A product is called a **multiple** of any of its factors.

Thus, abc is a multiple of a, of b, of c, of ab, of bc, and of ac.

Also, x^4 is a multiple of x, x^2, x^3, and x^4.

191. Common Multiple. An expression that is a multiple of two or more expressions is called a **common multiple** of these expressions.

Thus, $12x^3y^2$ is a common multiple of $3xy$ and $6x^3$.

192. Lowest Common Multiple. The lowest common multiple (l. c. m.) of two or more algebraic expressions is the algebraic expression of lowest degree that is divisible, without a remainder, by each of the given expressions, including both the numerical and literal parts.

For example, find the l. c. m. of $8a^2b$, $6abc$, and $-4ac^3$:

The literal common multiple of lowest degree is a^2bc^3. The least common multiple of 8, 6, and -4 is 24.

Thus, the lowest common multiple of $8a^2b$, $6abc$, and $-4ac^3$ is $24a^2bc^3$.

Although -24 is a common multiple of 8, 6, and -4, it is customary to take the lowest common multiple with the positive sign.

If the least common multiple is divided by each of the given expressions, the quotients will have no common divisor, other than unity.

193. *In the case of monomials,* the lowest common multiple is seen by inspection. Its numerical coefficient is the least common multiple of the given coefficients, and its literal part is the product of all the different letters, each with the highest exponent that it has in any of the given expressions.

194. *If expressions not monomials* are given, they must first be factored if possible, after which the factors of the lowest common multiple may be seen.

1. Find the l. c. m. of ax, $ac + ab$, and $cx^2 + bx^2$.

1. $ax = ax$.
2. $ac + ab = a(b + c)$.
3. $cx^2 + bx^2 = x^2(b + c)$.
4. $\therefore$ the l. c. m. is the product of a, x^2, and $b + c$, or $ax^2(b + c)$.

195. To find the l. c. m.: *Factor each expression into its prime factors. Then find the product of all the different prime factors, using each the greatest number of times it occurs in any of the given expressions.*

WRITTEN EXERCISES

Find the l. c. m. of:

1. $4\,ab$, $6\,ac$.
2. $5\,a^2$, $10\,ax$.
3. $6\,pr$, $9\,pq$.
4. $7\,x^2$, $3\,xy$.
5. xyz, yzw.
6. abc^2, a^2b^2c.
7. $x + y$, $ax + ay$.
8. $a^2 + ac$, $ab + bc$.
9. $bcx + bcy$, abc.
10. $13\,a^2 - 13\,b^2$, $39\,ab$.
11. $ax + xy$, $abc + bcy$.
12. pq, $apq - bpq$.
13. $a(b - c)$, $xb - xc$.
14. $17\,x^2$, $51\,y^2$, $17\,a^2$.
15. $(a + b)n$, $(a + b)r$.
16. $3\,a^2bc$, $5\,a^3b^2$, $15\,a^2b^3c$.
17. $(t - u)x$, $(t - u)xyz$.
18. $8\,xyz^2$, $24\,x^2y^2z$, $6\,xy^2z^2$.
19. $ax + x^2$, $ax - x^2$.
20. $a^3 + b^3$, $a^2 - b^2$.
21. $a - b$, $a + b$, $a^2 - b^2$.
22. $3\,r + 2$, $9\,r + 6$.
23. $x^2 - y^2$, $(x + y)^2$.
24. $(a - b)(b - c)$, $a^2 - b^2$.
25. $2(a - b)$, $2(a + b)$, $a^2 + b^2$.
26. $x^2 + y^2$, $x^2 - y^2$.
27. $1 - x$, $1 - x^2$.
28. $x^2 - 3\,x + 2$, $x - 2$.

29. $c + d,\ 2a - 3b$.

30. $x - 1,\ x + 1,\ x^2 - 1$.

31. $a^3 - b^3,\ a^2 + ab + b^2$.

32. $x^3 + y^3,\ x^2 - xy + y^2$.

33. $(x - y)^2,\ x^2 + 2xy + y^2$.

34. $c(a - b),\ a(b - a)$.

35. $(b - c)(c - a),\ (c - a)(a - b),\ (a - b)(b - c)$.

REVIEW

WRITTEN EXERCISES

Cancel the h. c. f. from the numerator and denominator of each fraction:

1. $\dfrac{(a^2 - b^2)x^2 - 2ax + 1}{ax - bx - 1}$.

2. $\dfrac{a^3 - 2a^2 - a + 2}{a^2 - 1}$.

3. $\dfrac{x^4 - 10x^2 + 9}{(x - 1)(x - 3)}$.

4. $\dfrac{y^4 - 5y^2 + 4}{y^2 + y - 2}$.

5. By factoring the expressions find the highest common factor of $x^4 + x^2y^2 + y^4$ and $x^3 + y^3$.

6. By factoring the expressions find the lowest common multiple of $x^2 - 3x + 2$, $x^2 - 1$, and $x^2 + 2x + 1$.

7. By factoring the expressions find the highest common factor of $(2x - 1)(x^3 - 1)$ and $(x^3 + x^2 + x)(x - 1)(x^2 - 1)$.

8. By factoring the expressions find the highest common factor and the lowest common multiple of the two expressions:

$$(x^2 - 1)(x^2 + 5x + 6),\ (x^2 + 3x)(x^2 - x - 6).$$

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define *an integral expression*. Sec. 180.

2. Define *common factor;* also *highest common factor*. Secs. 181, 185.

3. What is meant by *algebraically prime?* Sec. 183.

4. Define *multiple;* also *common multiple;* also *lowest common multiple*. Secs. 190–192.

5. State how to find the *h. c. f.* Sec. 189.

6. State how to find the *l. c. m.* Sec. 195.

CHAPTER XV

FRACTIONS

DEFINITIONS AND LAWS

196. Meaning of Fraction. In arithmetic the fraction $\frac{4}{12}$ is taken to mean either 4 of the 12 equal parts of a unit, the quotient of $4 \div 12$, or the ratio of 4 to 12.

All three questions:

What part of 12 *is* 4?
What is the quotient of $4 \div 12$?
What is the ratio of 4 *to* 12?

are answered by one fraction, $\frac{4}{12}$.

197. Fractions in Algebra. In algebra, similarly, the symbol $\frac{a}{b}$ stands for a of the b equal parts of a unit, or for the quotient of $a \div b$, or for the ratio of a to b; but it is usually regarded as an indicated division.

Symbols like $\frac{a}{b}$ and $\frac{a+x}{4q+p^2}$ are therefore usually read "a divided by b," and "$a + x$ divided by $4q + p^2$"; but for brevity, they may be read "a over b," and "$a + x$ over $4q + p^2$."

198. The dividend and the divisor of the indicated division are called the **numerator** and the **denominator** of the fraction; together they are called the **terms** of the fraction.

199. A fraction is said to be in its **lowest terms** when its numerator and denominator have no common factor.

200. The Sign of a Fraction. Every fraction, taken as a whole, has a sign before it, expressed or understood, in addition to the signs that the numerator and the denominator may contain.

201. The Law of Signs in Fractions. The sign of the quotient is changed if the sign of either the divisor or the dividend is changed (Sec. 77); hence,

To change the sign of either the numerator or the denominator is equivalent to changing the sign of the fraction.

Thus, if $\frac{2}{3}$ is changed to $\frac{-2}{3}$, the latter is the same as $-\frac{2}{3}$.

Similarly, if $\frac{2}{3}$ is changed to $\frac{2}{-3}$, the latter is the same as $-\frac{2}{3}$.

If the signs of both numerator and denominator are changed, the value of the fraction is unchanged.

Thus, $\frac{-a}{-b}=\frac{a}{b}$, also $\frac{b(-a)}{d(-c)}=\frac{-b(-a)}{-d(-c)}=\frac{a\cdot b}{c\,.\,d}$.

ORAL EXERCISES

State expressions equal to these, and having no negative signs in the numerator or denominator of the fractions:

1. $\frac{-3}{5}$.
2. $\frac{7}{-9}$.
3. $\frac{-4}{-7}$.
4. $\frac{-a}{x}$.
5. $\frac{4}{-d}$.
6. $\frac{-gt}{3}$.
7. $\frac{1}{-am}$.
8. $\frac{(-a)(-b)}{c}$.
9. $\frac{(-2)(3x)}{(-a)(-b)}$.
10. $\frac{-4}{(-3)5}$.
11. $\frac{-m}{p(-q)}$.
12. $\frac{2(-3x)}{a(-b)}$.
13. $\frac{3a(-2b)}{5t}$.
14. $\frac{4h}{(-3)(-5x)}$.
15. $\frac{(-2)(-3x)}{(-a)(-b)}$.
16. $\frac{-8x}{5(-3y)}$.

Express with denominator $x-y$:

17. $\frac{-7}{y-x}$.
18. $\frac{(-3)(-m)}{y-x}$.
19. $\frac{-a}{y-x}$.
20. $\frac{(-2a)(-5b)}{y-x}$.
21. $\frac{a-b}{y-x}$.
22. $\frac{36(a-b)}{y-x}$.

Express with numerator $m-r$:

23. $\frac{r-m}{2}$.
24. $\frac{r-m}{2x+3y}$.
25. $\frac{r-m}{-3q}$.
26. $\frac{r-m}{(-5x)(3y)}$.

27. How are the fractions $\frac{a+b}{a-b}$, $\frac{a+b}{b-a}$ related?

Compare similarly:

28. $\frac{b-a}{-2}$ and $\frac{a-b}{2}$; $\frac{b-a}{-c-d}$ and $\frac{a-b}{c+d}$.

29. $\frac{a(-b)(-c)}{ef}$ and $\frac{abc}{ef}$; $\frac{(-a)(-b)(-c)}{ef}$ and $\frac{abc}{ef}$.

WRITTEN EXERCISES

For each of the following write an equal fraction preceded by the sign + and having the same denominator as the original fraction:

1. $-\frac{-x}{y}$.
2. $-\frac{x}{-y}$.
3. $-\frac{5x}{3q}$.
4. $-\frac{-3m}{8}$.
5. $-\frac{1}{a}$.
6. $-\frac{-1}{b}$.
7. $-\frac{a-b}{2m}$.
8. $-\frac{a-b}{a+b}$.
9. $-\frac{3a+2b}{4x+5}$.
10. $-\frac{5m-1}{3+4x}$.
11. $-\frac{2-x^2}{x^2-3}$.
12. $-\frac{t^3+5}{1-t}$.

13. For each of the fractions in Exercises 1–12 write an equal fraction preceded by the sign + and having the same numerator as the given fraction.

202. If the numerator of a fraction is of the same degree as the denominator, or of higher degree, the fraction may be reduced to an integral expression or to an integral expression and a fraction (mixed expression).

To do this, *divide the numerator of the given fraction by the denominator.*

For example:

$$\frac{x^2+1}{x^2+x+1}=1-\frac{x}{x^2+x+1}.$$

$$\frac{20x^2-5x+3}{10x}=\frac{20x^2}{10x}-\frac{5x}{10x}+\frac{3}{10x}=2x-\frac{1}{2}+\frac{3}{10x}.$$

WRITTEN EXERCISES

Reduce to integral or mixed forms:

1. $\frac{at+at^2}{a}$.
2. $\frac{w-w^2}{w}$.
3. $\frac{2kt-k}{k}$.
4. $p\left(\frac{100+a}{100}\right)$.
5. $\frac{(t-tr)^2}{t^2}$.
6. $\frac{s(p+u)^2}{pu}$.
7. $\frac{25a^2b^2}{5ab}$.
8. $\frac{375x^3y^2}{25x^2y}$.
9. $\frac{a^2+b^2}{a+b}$.
10. $\frac{x^2+a^2}{x-a}$.
11. $\frac{a^2-3a}{a-2}$.
12. $\frac{36xy+5}{9x}$.
13. $\frac{2a^3+3a^2+1}{a}$.
14. $\frac{x^2+y+3}{x^2}$.
15. $\frac{4x^4+10x^2+5}{2x^2}$.

203. Mixed expressions are changed to the fractional form by reversing the process of Section 202. That is,

Multiply the integral part by the denominator of the fraction and add the product to the numerator of the fraction. The sum is the numerator of the result.

For example:

$$x^2+x+\frac{2}{x}=\frac{x(x^2+x)+2}{x}=\frac{x^3+x^2+2}{x}.$$

WRITTEN EXERCISES

Change each expression to a fraction:

1. $a+2b+\frac{c}{d}$.
2. $x+\frac{2y}{3z}$.
3. $a^2+\frac{3b}{5a^2}$.
4. $x^2+2xy+\frac{3}{y}$.
5. $x-1+\frac{5}{x+1}$.
6. $17x^4+3x^3+\frac{1}{x^2}$.
7. $a^2+2ab+b^2+\frac{c}{b}$.
8. $x^3+x^2+\frac{1}{x+1}$.
9. $x^2+2-\frac{1}{x^2-2}$.
10. $ay^2-3-\frac{y-4}{y^2}$.
11. $p^2-\frac{su}{pv}$.
12. $1-\frac{w^2}{x}$.
13. $p+\frac{pr}{100}$.
14. $vt+\frac{1}{v}$.
15. $3x-\frac{4x^2-2x-5}{x-1}$.

204. Principle of Reduction. Applying Section 128, *the value of a fraction is unchanged if both numerator and denominator are multiplied or divided by the same number.*

For example:

$$\frac{6}{2}=3 \text{ and } \frac{2\cdot 6}{2\cdot 2}=\frac{12}{4}=3.$$

$$\frac{24}{6}=4 \text{ and } \frac{24\div 3}{6\div 3}=\frac{8}{2}=4.$$

$$\frac{b}{c}=\frac{ab}{ac}; \text{ also } \frac{ax}{ay}=\frac{x}{y}.$$

205. To Reduce Fractions to Lowest Terms, *divide both numerator and denominator by all factors common to them, or by the h. c. f. of the numerator and denominator.*

The division may be indicated by canceling.

For example, $\frac{x\cancel{(a+x)}}{2y\cancel{(a+x)}}$ is reduced to $\frac{x}{2y}$ by dividing both numerator and denominator by $a+x$, their only common factor.

206. Law of Exponents. The law of exponents in division (Sec. 129) applies to fractions.

For example:

$$\frac{a^2}{a^5}=\frac{a\cdot a}{a\cdot a\cdot a\cdot a\cdot a}=\frac{a^2}{a^2\cdot a^3}=\frac{1}{a^3}; \text{ also } \frac{a^4b}{a^7c}=\frac{a^4b}{a^4a^3c}=\frac{b}{a^3c}.$$

$$\frac{x^4}{x^3}=\frac{x\cdot x\cdot x\cdot x}{x\cdot x\cdot x}=\frac{x^3\cdot x}{x^3}=x; \text{ also } \frac{14\,a^3b^2c}{7\,a^2bc}=\frac{2\cdot 7\,a^2bcab}{7\,a^2bc}=2\,ab.$$

$\frac{a^m}{a^r}$ means m factors, each a, in the numerator, and r factors, each a, in the denominator. They can be canceled from both numerator and denominator, one by one, until they are exhausted in either the numerator or the denominator.

ORAL EXERCISES

Reduce to lowest terms, and express without negative signs in the numerator or the denominator of the fraction itself:

1. $\frac{a^2}{a^5}$.
2. $\frac{3\,b^5}{4\,b^9}$.
3. $\frac{-x^7}{x^{13}}$.
4. $\frac{m^5}{m^2}$.
5. $\frac{-y^4}{2\,y^3}$.
6. $\frac{-t^5}{-t^3}$.
7. $\frac{-12\,p^3}{16\,p^5}$.
8. $\frac{20\,r^3}{-15\,r}$.

9. $-\dfrac{3c^3}{-2c^4}$.
11. $-\dfrac{-g^7}{4g^3}$.
13. $\dfrac{-x^{7n}}{x^{10n}}$.
15. $\dfrac{10^8}{10^{11}}$.

10. $-\dfrac{-a}{-a^4}$.
12. $-\dfrac{3a^2}{15a^8}$.
14. $\dfrac{a^{4n}}{a^{10n}}$.
16. $\dfrac{2^{4a}}{-2^{7a}}$.

WRITTEN EXERCISES

Reduce to lowest terms and express without using negative signs in the numerator or denominator of the fraction:

1. $\dfrac{-a^2bx}{-ab^2x^5}$.
2. $\dfrac{-a^2bc^n}{a^2c^2}$.
3. $-\dfrac{3ab^2c^2}{-6a^3bc^3}$.
4. $\dfrac{2^7a^{3p}}{2^4a^{2p}}$.
5. $\dfrac{m^{p+1}qv^2}{2qm}$.
6. $\dfrac{\frac{1}{2}mv^2}{m}$.
7. $\dfrac{mas}{am}$.
8. $\dfrac{-a^{3n}b^{7q}}{a^{6n}b^{5q}}$.
9. $\dfrac{m^pat}{m^n}$.
10. $\dfrac{-12x^2y^2z^{10}}{24x^4y^3z^5}$.
11. $\dfrac{a(-b^3)(-c^5)}{3a^2b(-c^2)}$.
12. $\dfrac{x^{n-1}y^{2n+2}}{x^{n-1}y^{n+2}}$.

207. Sometimes the common factors are made more apparent by factoring either the numerator or the denominator, or both.

For example:

$$\frac{ax-ay}{cx-cy}=\frac{a(x-y)}{c(x-y)}=\frac{a}{c}.$$

$$\frac{a^2-m^2}{4(a-m)}=\frac{(a+m)(a-m)}{4(a-m)}=\frac{a+m}{4}.$$

$$\frac{-5x-10a}{x^2+4ax+4a^2}=\frac{-5(x+2a)}{(x+2a)^2}=\frac{-5}{x+2a}=-\frac{5}{x+2a}.$$

WRITTEN EXERCISES

Reduce to lowest terms:

1. $\dfrac{a^2-b^2}{b-a}$.
2. $\dfrac{3x-3y}{7x-7y}$.
3. $\dfrac{ab-ac}{3b-3c}$.
4. $\dfrac{a^2-x^2}{4a+4x}$.
5. $\dfrac{1-y^2}{a+ay}$.
6. $\dfrac{a-b}{b-a}$.

7. $\frac{a-t}{a^2-t^2}$.

8. $\frac{a^2+ax}{ab+bx}$.

9. $\frac{x^2-2x+1}{x^2-x}$.

10. $\frac{a^2+2ap+p^2}{ab+bp}$.

11. $\frac{x^2-y^2}{y-x}$.

12. $\frac{d^2-9}{d^2+6d+9}$.

13. $\frac{(a^2-x^2)z}{(a-x)2y}$.

14. $\frac{a^2+b^2}{a^4-b^4}$.

15. $\frac{m^2-2mr+r^2}{m^3-r^3}$.

208. When several fractions have the same denominator, that denominator is called their **common denominator.** The common denominator must evidently be a multiple of the given denominators.

209. When the common denominator of several fractions is the l.c.m. of their denominators it is called the **lowest common denominator** (l.c.d.) of the given fractions.

210. *To reduce fractions to their lowest common denominator:*

Find the l.c.m. of their denominators for the l.c.d.

Divide the l.c.d. by the denominator of each fraction and multiply both terms of each fraction by the corresponding quotient.

WRITTEN EXERCISES

Change to fractions having the l.c.d.:

1. $\frac{2}{3x}, \frac{3}{4x^2}, \frac{1}{6x^3}$.

2. $\frac{a}{b}, \frac{b}{3c}, \frac{c}{2d}$.

3. $\frac{a}{2bx}, \frac{c}{abxy}, \frac{b}{3acx}$.

4. $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$.

5. $\frac{x^2}{2ab}, \frac{y^2}{3ac}, \frac{z^2}{4bc}$.

6. $\frac{a}{1-x}, \frac{1}{1-x^2}$.

7. $\frac{c}{a-c}, \frac{a}{a+c}$.

8. $\frac{4}{ax+x^2}, \frac{2}{ax-x^2}$.

9. $\frac{a+x}{a-x}, \frac{a-x}{a+x}$.

10. $\frac{4x^2}{3(a+b)}, \frac{xy}{6(a^2-b^2)}$.

11. $\frac{x^2}{a^2+b^2}, \frac{y^2}{a^2-b^2}$.

12. $\frac{1}{a-b}, \frac{1}{a+b}, \frac{abc}{a^2-b^2}$.

13. $\dfrac{3}{8(1-x)}, \dfrac{1}{8(1+x)}, \dfrac{x-1}{4(1+x^2)}.$

14. $\dfrac{1}{4\,a^3(a+b)}, \dfrac{1}{4\,a^3(a-b)}, \dfrac{1}{2\,a^2(a^2-b^2)}.$

15. $\dfrac{1}{2(x-y)}, \dfrac{1}{4(x+y)}, \dfrac{1}{6(x-y)^2}.$

16. $\dfrac{x}{(x-y)(y-z)}, \dfrac{y}{(x-y)(x-z)}, \dfrac{z}{(y-z)(x-z)}.$

ADDITION AND SUBTRACTION OF FRACTIONS

211. Preparatory.

1. What is the sum of $\frac{2}{3}$ and $\frac{1}{6}$? Of $\frac{1}{3}$ and $\frac{4}{5}$?

2. How must fractions be expressed before being added or subtracted?

3. What is the sum of $\dfrac{a}{b}$ and $\dfrac{c}{b}$? Of $\dfrac{x}{abc}$ and $\dfrac{y^2}{abc}$?

4. Subtract $\dfrac{ax}{bc}$ from $\dfrac{ay}{bc}$. Also, $\dfrac{m^2}{pq}$ from $\dfrac{n^2}{pq}$.

212. To Add or Subtract Fractions. 1. *Find the l. c. d. of the given fractions. This is the denominator of the result.*

2. *Reduce the given fractions to fractions having the l. c. d.*

3. *Find the algebraic sum of the numerators of the fractions resulting from step 2. This is the numerator of the result.*

4. *Reduce the results to lowest terms.*

EXAMPLES

1. Add $\dfrac{a}{a-x}$ and $\dfrac{3\,a}{a+x}$.

1. The l. c. d. is $(a-x)(a+x)=a^2-x^2$.

2. $\dfrac{a}{a-x}=\dfrac{a(a+x)}{a^2-x^2}=\dfrac{a^2+ax}{a^2-x^2}.$

3. $\dfrac{3\,a}{a+x}=\dfrac{3\,a(a-x)}{a^2-x^2}=\dfrac{3\,a^2-3\,ax}{a^2-x^2}.$

4. $\therefore\ \dfrac{a}{a-x}+\dfrac{3\,a}{a+x}=\dfrac{a^2+ax}{a^2-x^2}+\dfrac{3\,a^2-3\,ax}{a^2-x^2}=\dfrac{4\,a^2-2\,ax}{a^2-x^2}.$

2. Add $\frac{a}{b(c-a)(b-c)}, -\frac{b}{c(a-b)(a-c)}, -\frac{c}{a(a-b)(c-b)}$.

1. The l. c. m. is $abc\,(a-b)(b-c)(c-a)$.

Before dividing the l. c. m. by the given denominators we notice that $a-c$ in the second denominator is the same as $c-a$ in the l. c. m. with the signs changed. Also $c-b$ in the third denominator is the same as $b-c$ in the l. c. m. with the signs changed. Hence, the signs of each of these factors may be changed and by changing the signs before the second and third fractions, the denominators of the three fractions are now, $b(c-a)(b-c)$, $c(a-b)(c-a)$, and $a(a-b)(b-c)$, and the given fractions are all positive.

2. Dividing the l. c. m. by each denominator in turn we have $ac(a-b)$, $ab(b-c)$, and $bc(c-a)$.

3. Multiplying the results in step 2 by the numerators of the corresponding fractions and placing the sum over the l. c. m. we have the final sum:

$$\frac{a^2c\,(a-b)+ab^2\,(b-c)+bc^2\,(c-a)}{abc\,(a-b)(b-c)(c-a)}.$$

If the numerator and denominator of the sum have a common factor, it should be canceled so that the result will be in its lowest terms.

Test: Let $a=1$, $b=2$, $c=3$. Then the given fractions become

$$\frac{1}{2\cdot 2(-1)}-\frac{2}{3(-1)(-2)}-\frac{3}{1(-1)(+1)}=-\frac{1}{4}-\frac{1}{3}+3=\frac{29}{12}.$$

The result becomes $\frac{3(-1)+4(-1)+18(+2)}{6(-1)(-1)(+2)}=\frac{29}{12}$.

These being equal, we are reasonably sure that the work is correct.

If in checking the work in fractions with arbitrary values, the numbers chosen make any denominator zero, other values must be used.

WRITTEN EXERCISES

Add:

1. $\frac{a}{bx}, \frac{b}{x}$.
2. $\frac{a}{bc}, \frac{c}{xy}$.
3. $\frac{m}{n}, \frac{p}{q}$.
4. $\frac{3p}{4q}, \frac{p}{2q}$.
5. $\frac{1}{6z}, \frac{5}{12z}$.
6. $\frac{a}{x}, \frac{b}{y}$.
7. $\frac{1}{x^2}, \frac{5}{xy}$.
8. $\frac{b}{a}, \frac{w}{p}$.
9. $\frac{4}{1-x}, \frac{5}{1+x}$.
10. $\frac{2t-4}{(t+3)^2}, \frac{1}{t+3}$.
11. $\frac{c}{a}, \frac{c+x}{2a}$.
12. $\frac{a+x}{a-x}, \frac{3x+5a}{2(a-x)}$.

13. $\frac{1}{mn}, \frac{1}{pq}$.

14. $\frac{3}{a}, \frac{5}{a+1}$.

15. $\frac{1-p}{3v+2}, \frac{4p+5}{9v+6}$.

Subtract the second fraction from the first:

16. $\frac{3}{b}, \frac{2}{b}$.

17. $\frac{3a}{b}, \frac{2b}{a}$.

18. $\frac{1}{x}, \frac{1}{3x}$.

19. $\frac{5}{x}, \frac{a^2}{x^3}$.

20. $\frac{7}{a}, \frac{4}{ab}$.

21. $\frac{a}{b}, \frac{c}{d}$.

22. $\frac{4x}{a+5}, \frac{2x}{3(a+5)}$.

23. $\frac{7}{1-x}, \frac{5}{(1-x)^2}$.

24. $\frac{8}{m+1}, \frac{2}{m-1}$.

213. Signs before Fractions. Since the sign before the fraction relates to the fraction as a whole, the *whole* numerator is added or subtracted, as the case may be, the bar of the fraction having upon the numerator the effect of a parenthesis.

For example:

$$\frac{a}{c}-\frac{d-e}{c}=\frac{a-(d-e)}{c}=\frac{a-d+e}{c}.$$

$$\frac{a}{c}-\frac{d-e+f}{c}=\frac{a-(d-e+f)}{c}=\frac{a-d+e-f}{c}.$$

$$\begin{aligned}\frac{a}{x+1}+\frac{2a}{x-1}-\frac{5a-7ax}{x^2-1}&=\frac{ax-a}{x^2-1}+\frac{2ax+2a}{x^2-1}-\frac{5a-7ax}{x^2-1}\\&=\frac{ax-a+(2ax+2a)-(5a-7ax)}{x^2-1}\\&=\frac{10ax-4a}{x^2-1}.\end{aligned}$$

WRITTEN EXERCISES

Perform the operations indicated:

1. $\frac{5}{x}-\frac{4}{7x}-5x$.

2. $\frac{x-xy}{4}-\frac{x-xy}{2}$.

3. $\frac{x+y}{z}+\frac{x-2y}{2z}$.

4. $\frac{1}{x}-\frac{3-x}{x^2}$.

5. $\frac{x-2y}{5}+\frac{2x-y}{3}$.

6. $\frac{z}{2m^2q}+\frac{w}{2mq^2}$.

7. $\frac{3a^2b}{5x^2y}-\frac{5ab^2}{3xy}$.

8. $\frac{x}{ab}+\frac{y}{bc}-4xbc$.

9. $\frac{a}{c}-\frac{c}{d}-3ad+\frac{1}{2}c^2.$

10. $\frac{3a}{5x}-\frac{6b}{10y}+15xy.$

11. $\frac{a^2}{2xy}-\frac{4b^2+3a^2}{x^2}.$

12. $\frac{1}{a-b}+\frac{1}{a+b}.$

13. $\frac{a}{3}+\frac{a}{5}-\frac{a}{15}-5a.$

14. $\frac{a}{b^3c^2}+\frac{3b}{a^2c^3}+\frac{1}{ab^2c^2}.$

15. $\frac{a}{b}-\frac{b}{a}+\frac{b^2}{ab}+a^2b.$

16. $\frac{x^6}{x^4-a^4}-\frac{a^4x^2}{a^4+x^4}.$

17. $\frac{m}{q}+n-\frac{2n}{3q}.$

18. $\frac{x}{y}-\frac{w}{xy}-\frac{z}{wy}.$

19. $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}.$

20. $a+\frac{c}{6abxy}+\frac{b}{3acx}.$

21. $\frac{x-y}{x^2y}-\frac{x-y}{xy^2}.$

22. $\frac{a}{2b}-\frac{a-b}{2(a+b)}.$

23. $\frac{x}{2y}+\frac{x+y}{3(x-y)}.$

24. $\frac{a^2+b^2}{ab}-\frac{b^2+c^2}{bc}.$

25. $\frac{a+b}{a}+\frac{b+c}{b}+\frac{c+a}{c}.$

26. $\frac{a-b}{ab}-\frac{c-a}{ac}-\frac{b-c}{bc}.$

27. $\frac{1}{a-x}-3+\frac{2a}{(a+x)^2}.$

28. $\frac{1}{(x^2+1)^2}-\frac{x+1}{x^2+1}.$

29. $\frac{1}{2(x-1)}-\frac{1}{3(x+1)}-\frac{1}{x^2}.$

30. $\frac{2x^2-y^2}{x^2}-\frac{y^2-z^2}{y^2}-\frac{z^2-x^2}{z^2}.$

31. $\frac{1}{2(a-b)}+\frac{1}{2(a+b)}+\frac{a}{a^2+b^2}.$

32. $2-\frac{x^2-y^2}{x^2+y^2}+\frac{x^2+y^2}{x^2-y^2}.$

33. $\frac{1}{(a-b)(b-c)}+\frac{1}{a^2-b^2}.$

34. $\frac{x}{x^2-y^2}-\frac{y}{(x+y)^2}.$

35. $\frac{a}{c(a-b)}-\frac{c}{a(b-a)}.$ 36. $\frac{1}{(a-b)(c-a)}-\frac{1}{(b-a)(c-b)}.$

37. $\frac{a}{(b-c)(b-a)}+\frac{b}{(c-a)(c-b)}+\frac{c}{(a-b)(a-c)}.$

38. $\frac{a}{(b-c)(c-a)}+\frac{b}{(c-a)(a-b)}+\frac{c}{(a-b)(b-c)}.$

39. $\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}.$

40. $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}$.

41. $\frac{a(b+c)}{(a-b)(c-a)}+\frac{b(c+a)}{(b-c)(a-b)}+\frac{c(a+b)}{(c-a)(b-c)}$.

42. $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$.

43. $\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-z)(y-x)}+\frac{1}{z(z-x)(z-y)}$.

214. Preparatory.

Read as the sum or difference of two fractions:

1. $\frac{x+y}{3}$. 2. $\frac{a+5}{b}$. 3. $\frac{x+1}{x-1}$. 4. $\frac{4a+x}{a+9}$.

215. Separating Fractions into Parts. It is sometimes desirable to separate fractions into two or more addends, by reversing the process used in the addition of fractions.

For example:

1. $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$.

2. $\frac{ax+5y}{10ay}=\frac{ax}{10ay}+\frac{5y}{10ay}=\frac{x}{10y}+\frac{1}{2a}$.

3. $\frac{x^3+x^2+1}{x+1}=x^2+\frac{1}{x+1}$.

4. $$\frac{(a+x)^2-3(a-x)}{a^2-x^2}=\frac{(a+x)^2}{a^2-x^2}-\frac{3(a-x)}{a^2-x^2}=\frac{a+x}{a-x}-\frac{3}{a+x}$$
$$=\frac{a}{a-x}+\frac{x}{a-x}-\frac{3}{a+x}.$$

WRITTEN EXERCISES

Write each fraction as the algebraic sum of two or more fractions, reduce the fractions to lowest terms, and unite those with the same denominator:

1. $\frac{5x+8y}{10a}$.

2. $\frac{6a^2-3b^2}{14ab}$.

3. $\frac{a^2x+4b^2y}{12ay}$.

4. $\frac{a+x}{y}+\frac{a^2b-5dx}{aby}$.

5. $\frac{a+7b+c}{21a^2x}$.

6. $\frac{3g+5h}{30(g+1)}$.

7. $\frac{a^2-b^2+5}{a^2-2ab+b^2}$.

8. $\frac{(a-1)^2-11(a+1)}{a^2-1}$.

9. $\frac{9y-14x}{6xy}+\frac{15z-2ax}{6xz}-xyz$.

10. $\frac{7y-3x}{3y}-\frac{4ay-9ax+15y}{3ay}+6a^2y^2$.

11. $\frac{ab^2+8b+8a}{8ab}-\frac{6bc-18ac-6ab}{6abc}$.

12. $\frac{4x^2+4x+1-(1-3x-10x^2)}{10x+5}$.

13. $\frac{a^3-b^3+(2a^2-7ab+5b^2)}{3(a-b)}$.

14. $\frac{a^2-b^2-3(a-b)}{a+b}-\frac{7a^2+7a-(a^2-1)}{5(a+1)}$.

15. $\frac{4(1+r)+3r(1-r)}{1-r^2}-\frac{7a(1+r)-6r(1-r)}{a-ar^2}$.

MULTIPLICATION OF FRACTIONS

216. Multiplying Fractions. *The product of two or more fractions is the product of their numerators divided by the product of their denominators.*

For example:

$$\frac{2}{3}\cdot\frac{5}{7}=\frac{2\cdot 5}{3\cdot 7}=\frac{10}{21}.$$

$$\frac{m}{n}\cdot\frac{p}{q}=\frac{mp}{nq},\text{ and }\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{3e}{4}=\frac{3ace}{4bd}.$$

$$\frac{a}{c}\cdot\frac{-b}{d}=\frac{a\cdot -b}{c\cdot d}=\frac{-ab}{cd}=-\frac{ab}{cd}.$$

Since every integer or integral expression can be regarded as a fraction with denominator 1, the above definition of product also includes the case where one of the factors is an integral expression.

217. In simplifying the results of multiplication, canceling may be helpful.

For example:

$$\frac{4a}{7b}\cdot\frac{14b}{a^2}=\frac{4\cancel{a}}{\cancel{7}\cancel{b}}\cdot\frac{\overset{2}{\cancel{14}}\cancel{b}}{\cancel{a}a}=\frac{8}{a}.$$

If necessary, factor before canceling.

For example:

$$\frac{x^2}{34}\cdot\frac{51}{ax+x^2}=\frac{\cancel{x}^2}{2\cdot\cancel{17}}\cdot\frac{3\cdot\cancel{17}}{\cancel{x}(a+x)}=\frac{3x}{2(a+x)}.$$

$$\frac{x^2-4}{3+a}\cdot\frac{9+6a+a^2}{5x+10}=\frac{(x-2)\cancel{(x+2)}\cancel{(3+a)}^2}{5\cancel{(3+a)}\cancel{(x+2)}}$$
$$=\frac{(x-2)(3+a)}{5}.$$

Only common factors of the whole numerator and denominator may be canceled. *E.g.* in $\frac{5x-2}{5x+2}$ neither the 5 nor the x nor the 2 may be canceled; but in $\frac{5(x-2)}{5(x+2)}$ the 5 may be canceled. What is wrong with this indicated work, $\frac{\cancel{3}\cancel{a}+5b}{\cancel{a}-\cancel{3}c}$?

ORAL EXERCISES

Multiply:

1. $\frac{4b}{3x}\cdot\frac{1}{5}$.
2. $\frac{x^2}{5}\cdot\frac{x^3}{4}$.
3. $\frac{1}{a^3}\cdot\frac{1}{a^4}$.
4. $\frac{x^m}{3}\cdot\frac{x^n}{7}$.
5. $\frac{a}{4b}\cdot\frac{c}{3d}$.
6. $6a\cdot\frac{4b}{5c}$.
7. $p^3\cdot\frac{p^7}{4a}$.
8. $\frac{a^n}{b^{2n}}\cdot\frac{a^5}{b^8}$.

Multiply each of the following in turn by 12, and reduce to lowest terms:

9. $\frac{8}{3ax}$.
10. $\frac{4}{2by}$.
11. $\frac{5x}{ac}$.
12. $\frac{4y}{3x}$.
13. $\frac{2m}{6aby}$.
14. $\frac{7ax}{3b}$.
15. $\frac{8d}{12c}$.
16. $\frac{5t}{24s}$.

Multiply Exercises 9–16 above in turn by:

17. $3a$. 18. $2b$. 19. xy. 20. $4ax$. 21. $3cy$.

WRITTEN EXERCISES

Multiply:

1. $\frac{x^5}{a^4} \cdot \frac{x^7}{a^7}$.

2. $\frac{a^2b}{cd} \cdot \frac{3\,a^4}{c^3d}$.

3. $\frac{4\,x}{5\,a} \cdot \frac{2\,y}{3\,ab}$.

4. $\frac{a^{5n+1}}{10^{3n+2}} \cdot \frac{a^6}{10^{2n}}$.

5. $\frac{a^2b^{n+2}}{7\,c^3} \cdot \frac{a^{3n}b^5}{4\,d^{3n}}$.

6. $\frac{2^8}{y^5} \cdot \frac{x^{3n}}{y^7z^{3n}}$.

Multiply and reduce to lowest terms:

7. $\frac{7}{13\,a} \cdot \frac{39\,a^2}{49}$.

8. $\frac{6\,x^2}{a} \cdot \frac{a^2}{15\,x}$.

9. $\frac{a}{b} \cdot \frac{cb^2}{ad}$.

10. $\frac{p}{q} \cdot \frac{q^2}{3\,p^3}$.

11. $\frac{a}{3\,c} \cdot \frac{9\,c}{ad}$.

12. $\frac{x}{y^2} \cdot \frac{z^2}{w}$.

13. $z^2 \cdot \frac{xy^2z}{zw}$.

14. $\frac{x}{y} \cdot \frac{x+y}{5\,xy}$.

15. $\frac{p^2}{q^2} \cdot \frac{aq}{bp}$.

16. $\frac{m+n}{a} \cdot ab$.

17. $\frac{1}{c^2} \cdot \frac{bc}{(x+y)^2}$.

18. $\frac{x}{y} \cdot \frac{y^2}{z^2} \cdot \frac{z^3}{w^3}$.

19. $p^4q^4 \cdot \frac{ab}{p^3q^3}$.

20. $\frac{d^2c^2}{a^2b^2} \cdot \frac{ab}{cd}$.

21. $\frac{4\,b^2}{xy^2} \cdot xy$.

22. $\frac{mn}{3\,x} \cdot 3\,mx$.

23. $\frac{3\,c}{abx} \cdot ab$.

24. $\frac{a}{bx} \cdot \frac{cx}{d}$.

25. $\frac{2\,x}{a} \cdot \frac{3\,ab}{c} \cdot \frac{3\,ac}{2\,b}$.

26. $a \cdot \frac{1}{bc} \cdot \frac{c}{a}$.

27. $3\,bc \cdot \frac{b+c}{2\,bc}$.

28. $-x^2 \cdot \frac{-x}{ab}$.

29. $\frac{1}{x} \cdot \frac{x^2-xy}{6\,bc}$.

30. $\frac{-2\,a^2}{c^2} \cdot \frac{-b}{d}$.

31. $\frac{1}{a^3b^4xy^2} \cdot \frac{1}{abx^2y}$.

32. $\frac{x^3b^2z}{ab^2c} \cdot \frac{a^3b^2c}{xy^2z^4}$.

33. $\frac{3\,m^4n^3s}{4\,xy^2z^3} \cdot \frac{6\,x^4y^3z^2}{5\,m^2ns}$.

34. $\frac{ab^2}{cd^2} \cdot \frac{c^2d}{e^2f} \cdot \frac{ef}{ab}$.

35. $(x+3) \cdot \frac{x-2}{x-3}$.

36. $\frac{4\,x}{a+5} \cdot \frac{2(a+5)}{3\,x^3}$.

37. $\frac{am-bm}{c+1} \cdot \frac{2}{ar-br}$.

38. $(a^2-1) \cdot \frac{ax}{a+1}$.

39. $\frac{1}{2}xy^2\left(\frac{1}{x}+\frac{1}{y}\right)$.

40. $-\frac{3}{4}a^2b^2\left(\frac{a^2}{b^2}-\frac{a}{b}\right)$.

41. $abc^3\left(\frac{1}{a}-\frac{1}{b}\right)$.

42. $x^2y^2\left(\frac{1}{x^2}+\frac{1}{y^2}\right)$.

43. $\frac{1}{2}xy\left(\frac{x}{y}-\frac{y}{x}\right)$.

44. $\left(x+\frac{1}{y}\right)\left(x-\frac{1}{y}\right)$.

45. $\left(\frac{m}{n}+n\right)\left(\frac{m}{n}-n\right)$.

46. $\frac{x^2-2x+1}{q^3}\cdot\frac{3q^2}{x-1}$.

47. $\frac{1+6a+9a^2}{21}\cdot\frac{7t}{2+6a}$.

48. $\frac{4d^3}{x^2-10x+25}\cdot\frac{x^2-25}{8d}$.

49. $\frac{9-a^2}{a-4}\cdot\frac{16-8a+a^2}{a^2+6a+9}$.

50. $\frac{7x+14}{1-5a}\cdot\frac{2a-3}{(x+2)^2}\cdot\frac{x+2}{6a-9}$.

51. $\frac{a}{x-1}\cdot\frac{1-x^2}{2a+ay}\cdot\frac{2+y}{1+x}$.

52. $\frac{3m^2+2m}{6m^2-11m+3}\cdot\frac{3m^2+2m-1}{3m+2}$.

Simplify, but do not multiply the factors in the terms of the final result:

53. $\frac{x^2+2y^2-2xy}{a^4-b^4}\cdot\frac{a^2-2ab+b^2}{x^4+4y^4}$.

54. $\frac{c^2-d^2}{m^3-n^3}\cdot\frac{m^2+mn+n^2}{2ad-2ac+ce-ed}$.

55. $\frac{3x^2-xy}{x^2-x-12}\cdot\frac{2x^2-7x-4}{12x^2+11xy-5y^2}$.

56. $\frac{ax+an-am}{\frac{1}{x^2}+2+\frac{1}{y^2}}\cdot\frac{\frac{1}{y}+\frac{1}{x}}{x^2-2(m-n)+(m-n)^2}$.

57. $\frac{10bx+5by-20bz}{a+b+x}\cdot\frac{a^2-x^2+b^2+2ab}{8ax+4ay-16az}$.

58. $\frac{m^4-1}{(a+b+c)^2+ax+bx+cx}\cdot\frac{a^2+b^2+c^2+2ab+2ac+2bc}{p(m+1)^2+3m+3}$.

59. $\frac{a^4+4b^4}{c+2c^2-3c^3-cd}\cdot\frac{ab+2abc-3abc^2-abd}{a^2+2b^2-2ab}$.

218. Powers of Fractions. *Any power of a fraction equals that power of the numerator divided by the same power of the denominator.*

For example:

$$\left(-\frac{2}{3}\right)^2 = \frac{(-2)^2}{(3)^2} = \frac{4}{9}.$$

$$\left(\frac{a}{b}\right)^3 = \frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b} = \frac{a^3}{b^3}.$$

$$\left(\frac{a}{b}\right)^n = \frac{a}{b}\cdot\frac{a}{b}\cdots\text{ to } n \text{ factors.}$$

$$= \frac{a\cdot a\cdots\text{ to } n \text{ factors}}{b\cdot b\cdots\text{ to } n \text{ factors}} = \frac{a^n}{b^n}.$$

$$\left(\frac{3\,a^3x^2}{-5\,bc}\right)^3 = \frac{(3\,a^3x^2)^3}{(-5\,bc)^3} = \frac{27\,a^9x^6}{-125\,b^3c^3} = -\frac{27\,a^9x^6}{125\,b^3c^3}.$$

$$\left(\frac{m+r}{m-r}\right)^2 = \frac{(m+r)^2}{(m-r)^2} = \frac{m^2+2\,mr+r^2}{m^2-2\,mr+r^2}.$$

Reversing the order,

$$\frac{a^2+6\,a+9}{x^2-10\,x+25} = \frac{(a+3)^2}{(x-5)^2} = \left(\frac{a+3}{x-5}\right)^2.$$

WRITTEN EXERCISES

Write without parentheses:

1. $\left(\frac{a^3}{b^n}\right)^2.$
2. $\left(\frac{a^{4n}}{x^5}\right)^3.$
3. $\left(\frac{-2\,a\,3\,b^2}{5\,xy}\right)^3.$
4. $\left(1+\frac{x}{y}\right)\left(1+\frac{x}{y}\right).$
5. $\left(\frac{a}{b}\right)^{10}.$
6. $\left(\frac{a}{b}\right)^n.$
7. $\left(\frac{-3\,abc^2}{4\,m^3n}\right)^4.$
8. $\left(-\frac{2\,x-3}{12}\right)^3.$
9. $\left(\frac{a^2}{x^3}\right)^{3p}.$
10. $\left(\frac{1}{x^2}\right)^{q+1}.$
11. $\left(\frac{2\,ax}{3\,b^2y}\right)^5.$
12. $\left(\frac{2\,a^2b^3}{3\,x^3y}\right)^4.$
13. $\left(\frac{a}{x}-\frac{b}{y}\right)^2.$
14. $\left(\frac{-abx}{c^2dy}\right)^7.$
15. $\left(1+\frac{ab}{a+b}\right)\left(1+\frac{ab}{a+b}\right).$
16. $\left(\frac{p}{q}+\frac{q}{p}\right)\left(\frac{p}{q}+\frac{q}{p}\right).$
17. $\left(\frac{1}{a}+\frac{1}{b}\right)^2.$
18. $\left(\frac{a+bc}{a+b}\right)^2.$

19. $\left(\frac{1}{x}+\frac{1}{y}\right)^2$.

20. $\left(\frac{1}{m^2}+1\right)^2$.

21. $\left(\frac{x}{y}+\frac{y}{x}\right)^2$.

22. $\left(1+\frac{1}{x}\right)^2$.

23. $\left(\frac{1}{x}-1\right)^2$.

24. $\left(\frac{1}{x}-\frac{1}{y}\right)^2$.

25. $\left(\frac{x}{y}-\frac{y}{x}\right)^2$.

26. $\left(\frac{1}{m^2}-1\right)^2$.

27. $\left(\frac{1}{x}-x\right)^2$.

28. $\left(\frac{a}{b}+1\right)^2$.

29. $\left(\frac{c}{2}+a\right)^2$.

30. $\left(\frac{a}{b}-c\right)^2$.

Write each of the following as a power of a fraction:

31. $\frac{x^2-2x+1}{a^2+2a+1}$.

32. $\frac{a^2+2ab+b^2}{x^4-2x^3y+x^2y^2}$.

33. $\frac{x^2+4x+4}{4x^2+4x+1}$.

34. $\frac{a^2-4ab+4b^2}{9a^2-12ab+4b^2}$.

35. $\frac{a^6b^3}{x^3+3x^2+3x+1}$.

36. $\frac{100p^4-20p^2+1}{q^4-20q^2+100}$.

37. $\frac{a^3-9a^2+27a-27}{27x^3-27x^2+9x-1}$.

38. $\frac{x^2-2x+1}{x^4+2x^3y+x^2y^2}$.

DIVISION OF FRACTIONS

219. Reciprocal. If the product of two numbers is 1, each is called the **reciprocal** of the other.

Thus,

5 and $\frac{1}{5}$ are reciprocals of each other, because $5 \cdot \frac{1}{5} = 1$.

$\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals of each other, because $\frac{a}{b} \cdot \frac{b}{a} = 1$.

220. Division of Fractions. *To divide by a fraction multiply by the reciprocal of the fraction, that is, by the fraction inverted.*

Thus, to divide $\frac{4}{5}$ by $\frac{2}{3}$ multiply $\frac{4}{5}$ by $\frac{3}{2}$,

and to divide $\frac{a}{b}$ by $\frac{x}{3y}$ multiply $\frac{a}{b}$ by $\frac{3y}{x}$.

Test. If the quotient is correct, the product of the quotient and the divisor equals the dividend.

WRITTEN EXERCISES

Divide:

1. $\frac{1}{a} \div \frac{1}{3}$.
2. $\frac{1}{a} \div \frac{3}{a}$.
3. $\frac{1}{a} \div \frac{1}{b}$.
4. $\frac{3}{a} \div \frac{5}{b}$.
5. $\frac{a}{b} \div \frac{1}{b}$.
6. $\frac{m}{n} \div \frac{a}{b}$.
7. $\frac{a}{2b} \div \frac{c}{3d}$.
8. $\frac{m}{5n} \div \frac{5m}{21n}$.
9. $\frac{p}{7q} \div \frac{3p}{14q}$.
10. $\frac{n}{m} \div \frac{p}{q}$.
11. $\frac{x}{y} \div \frac{y}{x}$.
12. $\frac{1}{3ab} \div \frac{3}{ab}$.
13. $\frac{x^2}{y} \div \frac{x}{y^2}$.
14. $\frac{mn}{pq} \div \frac{m}{p}$.
15. $\frac{mx}{ny} \div \frac{m}{n}$.
16. $\frac{a^2}{b^2} \div \frac{a}{b}$.
17. $\frac{3a^2}{2b^2} \div \frac{2a}{3b}$.
18. $\frac{5x^2}{y^2} \div \frac{x^2}{y}$.
19. $\frac{15m^2}{n^2} \div \frac{5}{n^2}$.
20. $\frac{ab}{c} \div \frac{b}{c^2}$.
21. $\frac{6a}{b} \div \frac{a}{x}$.
22. $\frac{7a^2}{5b} \div \frac{14}{5ab^2}$.
23. $\frac{y^2z}{xw} \div \frac{yz}{x}$.
24. $\frac{m^2}{xyz} \div \frac{10m}{3x}$.
25. $\frac{a}{bx} \div \frac{d}{cx} = ?$
26. $\frac{5ax}{3cy} \div \frac{5x}{cy} = ?$
27. $\frac{b}{x-y} \div \frac{a}{x+y} = ?$
28. $\frac{3bx}{2az^a} \div \frac{ax}{2z^{b+a}} = ?$
29. $\frac{ac}{bd} \div \frac{ab}{cd} = ?$
30. $\frac{4a}{x^{m+1}} \div \frac{2b}{x} = ?$
31. $\frac{3ab}{5a^2c} \div \frac{6a^2b}{5a^2c^2} = ?$
32. $\frac{4a^2b^2}{15c^2d^2} \div \frac{4ab}{3c^2d} = ?$
33. $\frac{2x}{3y} \div \frac{4x}{3y} = ?$
34. $\frac{2x^2}{yz} \div \frac{3xyc}{z^2d} = ?$
35. $\frac{3abx}{5c^2} \div \frac{6a^2x^2}{10cb^2} = ?$
36. $\frac{2y}{3x} \div \frac{a}{b} \cdot \frac{9x^2}{4y^2} = ?$
37. $\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{e} \cdot \frac{e}{f} \div \frac{a}{f} = ?$
38. $\frac{a^{2n}b^{2n}c^{2n}}{xyz} \div \frac{a^nb^nc^n}{x^2y^2z^2} = ?$
39. $\frac{x^{2p}}{y} \cdot \frac{y^2}{x^p} \div \frac{x^3}{y^3} = ?$
40. $\frac{m^2n}{p^2q} \cdot \frac{n^2p}{q^2r} \div \frac{mnp}{qr} = ?$

221. Since every integral expression can be regarded as a fraction with denominator 1, the process of division includes also the case where one of the fractions is an integral expression.

For example:

$$-\tfrac{3}{5} \div 6 = -\tfrac{3}{5} \div \tfrac{6}{1} = -\tfrac{3}{5} \cdot \tfrac{1}{6} = -\tfrac{1}{10}.$$

$$\frac{a}{-b} \div c = \frac{a}{-b} \div \frac{c}{1} = \frac{a}{-b} \cdot \frac{1}{c} = \frac{a}{-bc} \text{ or } -\frac{a}{bc}. \qquad \text{Sec. 201.}$$

$$\frac{3a^2x}{by} \div (-3a) = \frac{3a^2x}{by} \cdot \frac{1}{-3a} = -\frac{ax}{by}. \qquad \text{Sec. 201.}$$

$$x^2y \div \frac{-x}{z} = \frac{x^2y}{1} \cdot \frac{-z}{x} = -xyz.$$

WRITTEN EXERCISES

Perform the divisions indicated and reduce the results to lowest terms:

1. $\dfrac{3a^2x}{by} \div 2c.$
2. $\dfrac{2a}{b^2c} \div \dfrac{3a^2b}{c}.$
3. $\dfrac{-3a^2}{bx^2} \div a^2.$
4. $\dfrac{a^3+b^3}{a^2-b^2} \div \dfrac{a+b}{a-b}.$
5. $\dfrac{2ab}{3c^2} \div \dfrac{2b}{-a^3c}.$
6. $\dfrac{9x^2y}{12mn^2} \div \dfrac{3xy^2}{4m^2n}.$
7. $\dfrac{-ba^2}{7cb^2} \div \dfrac{3a}{21c^2b}.$
8. $\dfrac{-7x^3y}{12^2} \div \dfrac{7xy}{-4zw}.$
9. $\dfrac{5ab^2}{-21cd} \div \dfrac{-5a^2b}{7c^3d^2}.$
10. $\dfrac{-15mn^3}{25a^2b} \div \dfrac{3m^2n^2}{5ab}.$
11. $\dfrac{8a^2}{a^2b^2} \div \dfrac{4a}{a-b}.$
12. $\dfrac{a-x}{y} \div \dfrac{a^2-x^2}{-xy}.$
13. $\dfrac{x^3+y^3}{x(x-y)} \div \dfrac{x^2+2xy+y^2}{x^2-y^2}.$
14. $\dfrac{2a+3b}{c+d} \div \dfrac{c-d}{2a-3b}.$
15. $\dfrac{a}{a+x} \div \dfrac{a^2-a}{a+x}.$
16. $\dfrac{4x^2-9y^2}{a^2-b^2} \div \dfrac{2x+3y}{a-b}.$
17. $\dfrac{4x^2}{3(a+b)} \div \dfrac{xy}{6(a^2-b^2)}.$
18. $\dfrac{a^2}{a^2-b^2} \div \dfrac{2ab}{(a-b)^2}.$

19. $\dfrac{a^3-x^3}{a+x} \div \dfrac{a-x}{-(a+x)^2}$.

20. $\dfrac{x^2-b^2}{-bc} \div \dfrac{b+c}{-b(x+b)^2}$.

21. $\dfrac{2\,ax-x^2}{a(x+a)} \div \dfrac{x^2-a^2}{x+a}$.

22. $\dfrac{x^2-1}{x^2-3\,x+2} \div \dfrac{x-1}{x-2}$.

23. $\dfrac{x^2-2\,xy-3\,y^2}{x^2+2\,xy+y^2} \div \dfrac{x-3\,y}{x+y}$.

24. $\dfrac{a^2-ay}{m-n} \div \dfrac{a^4-y^4}{-(a-y)^2}$.

25. $\dfrac{3\,x^2-8\,x+4}{3\,x^2+2\,x-5} \div \dfrac{15\,x^2+11\,x-14}{12\,x^2+11\,x-15}$.

COMPLEX FRACTIONS

222. We have so far considered *simple* fractions, those in which neither the numerator nor the denominator is in fractional form.

Since a fraction indicates division, the division of two fractions may be indicated in fractional form.

Thus, $\frac{2}{3} \div \frac{7}{5}$ may be written $\dfrac{\frac{2}{3}}{\frac{7}{5}}$,

and $\dfrac{a}{b} \div \dfrac{c}{d}$ may be written $\dfrac{\dfrac{a}{b}}{\dfrac{c}{d}}$.

223. Complex Fractions. A fraction whose numerator or denominator, or both, are fractional expressions is called a **complex fraction.**

For example:

$$\dfrac{\dfrac{a+b}{c}}{d},\quad \dfrac{\dfrac{2}{7}}{3},\quad \dfrac{\dfrac{31}{2+a}}{\dfrac{x+2}{3}},\quad \dfrac{\dfrac{3\,x}{4\,y}}{\dfrac{7\,x}{12\,y}},$$ are complex fractions.

224. A complex fraction may be reduced to a simple fraction either by performing the indicated divisions, or by multiplying both numerator and denominator by the l. c. m. of their respective denominators. In each case any operations indicated in the numerator or the denominator should be performed first, as far as possible.

EXAMPLES

1. Simplify: $\dfrac{\dfrac{x^2-y^2}{a+b}}{\dfrac{x+y}{a^2-b^2}}$.

$$\frac{\dfrac{x^2-y^2}{a+b}}{\dfrac{x+y}{a^2-b^2}}=\frac{x^2-y^2}{a+b}\div\frac{x+y}{a^2-b^2}=\frac{x^2-y^2}{a+b}\times\frac{a^2-b^2}{x+y}=(x-y)(a-b).$$

2. Simplify: $\dfrac{\dfrac{a}{ax}+\dfrac{b}{ax}}{\dfrac{b+c}{by}}$.

$$\frac{\dfrac{a}{ax}+\dfrac{b}{ax}}{\dfrac{b+c}{by}}=\frac{\dfrac{a+b}{ax}}{\dfrac{b+c}{ay}}=\frac{\dfrac{axy(a+b)}{ax}}{\dfrac{axy(b+c)}{ay}}=\frac{y(a+b)}{x(b+c)}.$$

WRITTEN EXERCISES

Reduce to simple fractions:

1. $\dfrac{\dfrac{24x}{y}}{6x}$.

2. $\dfrac{\dfrac{4}{8}}{a^2}$.

3. $\dfrac{\dfrac{7a}{5c}}{21b}$.

4. $\dfrac{\dfrac{m^2-9}{m+3}}{a}$.

5. $\dfrac{\dfrac{1}{a-b}}{a+b}$.

6. $\dfrac{\dfrac{12ab}{35cd}}{\dfrac{25d}{16ac}}$.

7. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{x-y}$.

8. $\dfrac{\dfrac{1-49t^2}{7t+1}}{3ab}$.

9. $\dfrac{\dfrac{x^3-1}{5a}}{\dfrac{x-1}{15a^2b}}$.

10. $\dfrac{2+ax+\dfrac{1}{ax}}{a^2x^2-1}$.

11. $\dfrac{\dfrac{1}{x-y}-\dfrac{1}{x+y}}{\dfrac{y}{x-y}}$.

12. $\dfrac{\dfrac{2ax}{a+x}-a}{\dfrac{1}{x}+\dfrac{1}{a-2x}}$.

13. $\dfrac{\dfrac{a-3}{3b}-\dfrac{3b}{a-3}}{\dfrac{1}{3b}-\dfrac{1}{a-3}}.$

15. $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}.$

14. $\dfrac{\dfrac{a}{b^2}+\dfrac{b}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ab}+\dfrac{1}{b^2}}.$

16. $\dfrac{\dfrac{a^2-3ab}{x^2-1}}{\dfrac{9b^2-6ab+a^2}{1+x}}.$

17. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{a}{b}+\dfrac{b}{a}-1}-\dfrac{1+\dfrac{b^2}{a^2}-\dfrac{b}{a}}{\dfrac{a}{b}+\dfrac{b^2}{a^2}}.$

18. $\dfrac{\dfrac{ab}{c}+\dfrac{bc}{a}+\dfrac{ca}{b}}{\dfrac{a}{bc}+\dfrac{b}{ca}+\dfrac{c}{ab}}\cdot\left[\dfrac{(a+b+c)^2}{ab+bc+ca}-2\right].$

19. $\dfrac{x^4+x^3-x-1}{1-y^2}\cdot\dfrac{y^2-1}{x^2-x}\cdot\left[1-\dfrac{1}{1-\dfrac{1}{x}}\right].$

REVIEW

WRITTEN EXERCISES

Reduce to an integral or mixed expression:

1. $\dfrac{x^2+b}{x^2+c}.$ 2. $\dfrac{6x^2+5}{x+5}.$ 3. $\dfrac{x^3-a^3}{x-a}.$

Reduce each expression to a fraction:

4. $a+\dfrac{b}{c}.$ 5. $x+1-\dfrac{1}{x+1}.$ 6. $c+\dfrac{b^2}{a+b}.$

Reduce to lowest terms:

7. $\dfrac{x^2+2xy+y^2}{x^2-y^2}.$ 8. $\dfrac{9x^2-12x+4}{9x^2-4}.$

9. $\dfrac{a^2-4}{a^2-4a+4}$.

10. $\dfrac{8ax+4ay-16az}{10bx+5by-20bz}$.

11. $\dfrac{x^3-1}{x^2-1}$.

12. $\dfrac{(a+b)^2-c^2}{ax+bx+cx}$.

13. $\dfrac{a^2-x^2+b^2+2ab}{a+b+x}$.

14. $\dfrac{x^4+3x^3+x+3}{2x+6}$.

Add:

15. $\dfrac{a}{x}-\dfrac{1}{2x}$.

16. $\dfrac{x}{12a}-\dfrac{y}{4}$.

17. $\dfrac{a-2b}{c}+\dfrac{3b}{2c}$.

18. $\dfrac{x-2y}{8y}-\dfrac{2x-y}{12y}$.

19. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.

20. $\dfrac{1}{1+x}+\dfrac{1}{1-x}-\dfrac{2}{1-x^2}$.

21. $\dfrac{1}{x+a}+\dfrac{2}{x+b}-\dfrac{3}{x+c}$.

22. $\dfrac{5x+4}{x-2}-\dfrac{3x-2}{x-3}-\dfrac{x^2-2x-17}{x^2-5x+6}$.

23. $\dfrac{a-b}{a^2-ab+b^2}+\dfrac{1}{a+b}+\dfrac{ab}{a^3+b^3}$.

24. $\dfrac{(x+y)(x^2+y^2-1)}{xy}+\dfrac{(y+1)(y^2-x^2+1)}{y}+\dfrac{(x+1)(x^2-y^2+1)}{x}$.

25. $\dfrac{4}{x^2-y^2}+\dfrac{5}{x^3-y^3}$.

26. $\dfrac{x}{x^4-1}-\dfrac{x}{x^2+1}$.

27. $\dfrac{a}{a^4+a^2+1}+\dfrac{1}{a^2+a+1}$.

28. $\dfrac{x-3}{x^2-7x-18}+\dfrac{2x-1}{x^2-8x-9}$.

Multiply:

29. $\dfrac{x-1}{x}\cdot\dfrac{x+1}{-y}$.

30. $\dfrac{5ax}{3cy}\cdot\dfrac{xy-y^2}{x^2-xy}$.

31. $\dfrac{ax}{(a-x)^2}\cdot\dfrac{a^2-x^2}{-ab}$.

32. $\dfrac{x^2-2xy+x^2}{x+y}\cdot\dfrac{1}{x^2-y^2}$

33. $\dfrac{x+2}{x-1}\cdot\dfrac{x^2-1}{x^2-4}$.

34. $\dfrac{a+b}{-b}\cdot\dfrac{ac}{a^2-b^2}$.

35. $\frac{x^2-9}{x^2+4x}\cdot\frac{x^2-16}{x^2-3x}.$

36. $\frac{4x^2}{3(a+b)}\cdot\frac{6(a^2-b^2)}{-xy}.$

37. $\frac{-4ab}{6c}\cdot\frac{9c^2d}{-7a^2b}\cdot\frac{-21c}{12abc}.$

38. $\frac{a-b}{a^2+2ab}\cdot\frac{a^2-4b^2}{a^2-ab}.$

39. $\frac{x^3-8y^3}{x^2-y^2}\cdot\frac{x^2-xy-2y^2}{x^2-4xy+4y^2}.$

40. $\frac{4m+r}{25a^2-9}\cdot\frac{10am-6m}{16m^2+8mr+r^2}.$

41. $\frac{x^2+6x+5}{a^3+10a^2b+25ab^2}\cdot\frac{a^2-25b^2}{x^2+2x+1}.$

42. $\frac{x^2+6ax+5a^2}{x^2+2ax+a^2}\cdot\frac{ab+bx}{5ac+cx}.$

Divide:

43. $\frac{a^2}{b}\div\frac{a}{b^2}.$

44. $\frac{ac}{bd}\div\frac{ad}{bc}\div\frac{x^2}{y}\div\frac{x}{y^2}.$

45. $\frac{a+b}{a-b}\div\frac{(a+b)^2}{a-b}.$

46. $\frac{x^2+5x+6}{x^2-1}\div\frac{x+5}{x+1}.$

47. $\frac{a^2bc}{x^3-1}$ by $\frac{a^4bc^3}{x-1}.$

48. $\left(\frac{ax}{by}\right)^2\div\left(\frac{cy}{bx}\right)^3.$

49. $\left(\frac{a-b}{x-y}\right)^3\div\left(\frac{a^2-b^2}{x^2-y^2}\right)^2.$

50. $\left(\frac{m-n}{m-p}\right)^4\div\left(\frac{n-m}{n-p}\right)^4.$

51. $\frac{x}{1+x}\div\frac{x+x^2}{(1+x)^2}.$

52. $\frac{a^2-16}{p+t}$ by $\frac{a^2-4a}{pt+t^2}.$

53. $\frac{a^2-3a-28}{x^4+4y^4}$ by $\frac{a^2-7a}{x^2+2xy+2y^2}.$

54. $\frac{a^4+10a^2x^2+25x^4}{5x^2}$ by $\frac{a^4+7a^2x^2+10x^4}{10x^2-35x^3}.$

Simplify:

55. $1-\cfrac{1}{a-\cfrac{1}{b-\cfrac{1}{c}}}.$

56. $a-\cfrac{b}{c-\cfrac{d}{e-\cfrac{f}{g}}}.$

57. $\dfrac{\dfrac{x^2-y^2}{a^3+b^3}}{\dfrac{x^2-2xy+y^2}{a^2+2ab+b^2}}.$ 58. $\dfrac{\dfrac{15(x-2y)}{m^2+n^2}}{\dfrac{5x^2-10xy}{m^4-n^4}}$

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. State three meanings of a fraction. Sec. 196.

2. Which meaning of a fraction is commonly used in algebra? Sec. 197.

3. When is a fraction in its *lowest terms?* Sec. 199.

4. What is meant by the sign of a fraction? Sec. 200.

5. State the *Law of Signs* of fractions. Sec. 201.

6. When may the numerator be divided by the denominator? Sec. 202.

7. How may a mixed expression be changed to a fraction? Sec. 203.

8. State the principle that applies to the reduction of fractions. Sec. 204.

9. How is a fraction *reduced to its lowest terms?* Sec. 205.

10. What law of exponents applies to fractions? Sec. 206.

11. Define *common denominator;* also *lowest common denominator.* Secs. 208, 209.

12. How may fractions be changed to equivalent fractions having the lowest common denominator? Sec. 210.

13. How are fractions added? Multiplied? Divided? Secs. 212, 216, 220.

14. How is a power of a fraction obtained? Sec. 218.

15. What are *complex fractions* and how are they simplified? Secs. 223, 224.

CHAPTER XVI

EQUATIONS

PROPERTIES

225. Degree of an Equation. The degree of an equation is stated with respect to its unknowns. It is the highest degree to which the unknowns occur in any term in the equation. Unless otherwise stated, all the unknowns are considered.

For example:

1. $3x + 1 = 0$, and $4x + 7y - 3z = 6$, are equations of the first degree.
2. $4y^2 - y + 3 = 0$, and $x^2 + y^2 - 4 = 0$, are equations of the second degree.
3. $x^3 - 1 = 0$, $x^3 + 2y^2 - 4x + 3 = 0$, $5xyz + x^2 = 2y$, and $x^3 - 3y^3 + z = 5x$ are equations of the third degree.
4. $v = \frac{4\pi R^3}{3}$ is of the first degree in v and of the third degree in R.

226. An equation of the first degree is called a **linear equation.**

227. An equation of the second degree is called a **quadratic equation.**

228. An equation of the third or higher degree is called a **higher equation.**

It is unnecessary to define here the degree of expressions containing radicals.

229. In order to state the degree of an equation its terms must be united as much as possible.

Thus, $x^2 + 2x + 1 = x^2$ appears to be a quadratic equation.
But $2x + 1 = 0$, to which it reduces, is a linear equation.

230. Terms not involving the unknowns are called **absolute terms.**

Thus, in $x^2 + 5 = 3x - 2a$, 5 and $-2a$ are absolute terms.

231. Fractional Equations. An equation in which the unknown quantity occurs in any denominator is called a *fractional equation.*

Thus, $\frac{2}{x} - 3x + 1 = 0$ is a fractional equation ; also $\frac{1}{x^2} - \frac{a}{x+1} = b$.

We shall not define here the degree of fractional equations. It will be shown later that the process of clearing of fractions may change the degree of an equation.

ORAL EXERCISES

State the degree of each of the following equations with respect to each unknown:

1. $x + 2 = 0$.
2. $ax + 2 = 0$.
3. $z^2 - mz = 4$.
4. $x^2 + xy = 5$.
5. $\frac{x^2 - 4}{2} = 0$.
6. $\frac{1}{2}mv^2 = 6$.
7. $gt^2 = 32$.
8. $at + \frac{1}{2}gt^2 = 0$.
9. $ax^2 + b = 0$.
10. $ax^3 - by^2 = 1$.
11. $a^2x^2 + b^2y^2 = a^2b^2$.
12. $\frac{4}{3}\pi r^3 = 100$.
13. $\frac{w^2 - 1}{4} = 6$.
14. $2\pi r^2h = 50$.
15. $mv^2 - 16 = 0$.
16. $xy^2 - x^2y = 6$.

Select the linear, the quadratic, and the higher equations from the following:

17. $x + 3 = 0$.
18. $x^2 = 9$.
19. $x^3 = 27$.
20. $2x - 6 = 0$.
21. $ax = b + c$.
22. $mx^2 + px = q$.
23. $x^3 - x^2 - x = 3$.
24. $ax^2 + ax = c$.
25. $2x - 8x = 0$.
26. $3x^2 - x^2 + 3x = x^2 + 1$.
27. $2x^3 = 2x^2 + x + 2x^3$.
28. $x^4 + 5 = 3$.
29. $ay^3 - by^2 + 1 = 0$.
30. $5y^2 + 2y = 6 + 5y^2$.

SOLUTION OF LINEAR EQUATIONS

232. To Solve a Linear Equation with One Unknown. In general:

1. *Clear it of fractions, if there are any.*
2. *Remove the parentheses, if there are any.*
3. *Transpose all the terms containing the unknown quantity to one member, preferably the left, and all the other terms to the other member.*
4. *Unite the terms in each member as much as possible.*
5. *Divide both members by the coefficient of the unknown.*

EXAMPLE

Solve:
$$\frac{2x-5}{6}+\frac{6x+3}{4}=5x-\frac{35}{2}. \tag{1}$$

Multiplying each term by 12, the l. c. m. of the denominators,
$$2(2x-5)+3(6x+3)=60x-6\cdot 35. \tag{2}$$

Removing the parentheses,
$$4x-10+18x+9=60x-210. \tag{3}$$

Transposing -10, 9, and $60x$,
$$4x+18x-60x=10-9-210. \tag{4}$$

Uniting terms, and multiplying both members by -1,
$$38x=209. \tag{5}$$

Dividing both members by 38,
$$x=\tfrac{11}{2}. \tag{6}$$

TEST.
$$\frac{2(\frac{11}{2})-5}{6}+\frac{6(\frac{11}{2})+3}{4}=5(\tfrac{11}{2})-\frac{35}{2}.$$

WRITTEN EXERCISES

Solve and test:

1. $2(x+3)-3(x+2)=6(x+5).$
2. $x(x-1)-(2x-1)=x(x+6).$
3. $2x-\dfrac{11+x}{2}=\dfrac{19+x}{3}.$
4. $(x+3)^2=-7+(5-x)^2.$
5. $\dfrac{z-1}{2}+\dfrac{z-2}{3}-6=\dfrac{z-3}{4}.$
6. $\dfrac{x}{4}+\dfrac{4x-12}{5}-\dfrac{9x+2}{20}=0.$

7. $\frac{y}{12}-\frac{8-y}{8}+2\frac{3}{4}=\frac{5+y}{4}.$

8. $5(x-4)+4(x-3)-x(x-1)=x(3-x).$

9. $\frac{1}{2}(x-5)-\frac{1}{3}(x-4)=\frac{1}{2}(x-3)-(x-2).$

10. $\frac{y}{2}+\frac{y}{4}-\frac{y}{6}=7.$

11. $\frac{y}{12}-\frac{3-y}{8}=\frac{5+y}{4}-2\frac{3}{4}.$

12. $\frac{x}{3}-\frac{x}{4}=\frac{1}{2}+\frac{x}{5}-\frac{x}{6}.$

13. $\frac{x-1}{2}+\frac{x-2}{3}=6+\frac{x-3}{4}.$

14. $\frac{y}{3}-\frac{y}{4}-\frac{1}{2}=\frac{3y}{4}+1.$

15. $\frac{2x}{3}+\frac{5x}{9}=\frac{x}{6}+\frac{x}{2}+10.$

16. $\frac{x}{5}-\frac{x-8}{4}=\frac{x}{20}.$

17. $\frac{7}{10}x+\frac{2}{5}(20-x)=10.$

18. $\frac{1}{3}(x-3)-\frac{1}{5}(x-5)=\frac{1}{15}(x-15)+5.$

233. A fraction that is not given in its lowest terms should first be reduced.

EXAMPLE

Solve: $x-\frac{x^2-1}{x+1}=2x-3.$

Simplifying the fraction, $x-(x-1)=2x-3.$

Thus, $x-x+1=2x-3,$

and, $x=2.$

TEST. $2-\frac{2^2-1}{2+1}=1=2.2-3.$

But suppose the given equation to be solved by clearing of fractions instead of by first reducing the fraction:

Multiplying by the l. c. d., $x+1$,

$$x(x+1)-(x^2-1)=(2x-3)(x+1).$$

Removing parentheses, $2x^2-2x-4=0.$

Dividing by 2, $x^2-x-2=0.$

Factoring, $(x-2)(x+1)=0.$

Then, $x-2=0$ and $x+1=0,$

and, $x=2$ and $x=-1.$

In this way a new value, -1, is introduced which is not a root of the given equation (see Sec. 172).

WRITTEN EXERCISES

Solve and test:

1. $\frac{x-2}{x^2-4}=\frac{2}{x-2}.$

2. $\frac{x^2-3}{x-1}=x+2.$

3. $\frac{1}{3}=x-\frac{x^2+1}{x-1}.$

4. $\frac{y}{2}-\frac{y^2-1}{y+1}=y.$

5. $\frac{5}{x^2}-\frac{1}{x}=-\frac{2x-1}{x^2}.$

6. $\frac{1}{x-1}+\frac{1}{x+1}=\frac{4}{x^2-1}.$

7. $\frac{2}{w-2}-\frac{4}{w+2}=\frac{7}{w^2-4}.$

8. $\frac{3}{x}-\frac{5}{x+6}=\frac{1}{x^2+6x}.$

9. $\frac{x+1}{x-1}=\frac{x}{(1-x^2)x}+\frac{x}{x-1}.$

10. $\frac{3x}{x(x+2)}=\frac{5}{x-5}.$

11. $\frac{(6x-2)x}{5x^2-8x}=4.$

12. $x-\frac{8x}{3x^2}=\frac{2x^2-x}{2x}.$

13 $\frac{1}{x-2}-\frac{x}{x+2}=\frac{4-x}{x+2}.$

14. $\frac{2}{y}+\frac{y}{y^2-y}=\frac{3}{y(y-1)}.$

15. $\frac{2y+1}{y-3}+5=\frac{y^2-7y}{(y-3)y}.$

16. $\frac{8}{2x-8}-\frac{5x}{3x-12}+5=0.$

17. $\frac{3x}{x+5}-\frac{6}{2x+10}=\frac{7x+9}{3x+15}.$

18. $\frac{3}{4}-\frac{2x-5}{3x}+\frac{(x-3)x}{4x^2}=0.$

19. $x+2=\frac{1}{x+1}+\frac{x^2-25}{x+5}.$

20. $\frac{1}{y-3}+\frac{3}{y+3}=\frac{6y+18}{y^2-9}.$

21. $\frac{w+1}{w-4}+\frac{w+2}{w+7}=\frac{2w^2-3w+5}{w^2+3w-28}.$

22. $\frac{x-2}{x+3}-\frac{x+2}{x+5}=\frac{4x-1}{x^2+8x+15}.$

234. Decimal Coefficients. Equations in fractional form, like those in Secs. 231–233 should be distinguished from equations, having decimal coefficients. The former present peculiar difficulties, as explained above, but the latter yield to the same processes as equations with integral coefficients.

If equations fractional in themselves contain decimals, the processes of Secs. 232, 233 are to be applied first.

235. PREPARATORY.

We have already used decimal coefficients in evaluating, in adding, in subtracting, and in solving equations containing per cents.

Example: A number less 5 % of itself is 4.75. What is the number?

SOLUTION. 1. $x - .05\,x = 4.75.$

2. $.95\,x = 4.75.$

3. $x = \frac{4.75}{.95} = \frac{475}{95} = 5$, the number.

This equation could be written $x - \frac{5}{100}x = 4\frac{75}{100}$, but this is not so simple.

236. *In simplifying equations with decimal coefficients, it is generally better to work with the decimals or multiply both members by a power of 10 so as to make all the decimals whole numbers, instead of substituting for them their common fractional equivalents.*

EXAMPLE

A part of $100 was lent at 5 % annually, and the rest at 7 % annually; the total interest for one year was $5.50. What was each sum?

SOLUTION. 1. $.05\,x + .07(100 - x) = 5.50.$

2. $5\,x + 7(100 - x) = 550.$ Multiplying (1) by 100.

3. $\therefore 2\,x = 150$, and $x = 75.$

4. The parts were $75 and $25.

WRITTEN EXERCISES

1. $.5\,x + 8 = 20.5.$

2. $.05\,x - 6.25 = -3.75.$

3. $.15\,x + .03\,x = .25\,x - 2.1.$

4. $3.5\,x - 1.6\,x = 3\,x - .75.$

5. $1.01\,x + 2.005 - .003\,x = 45.306.$

6. $4.2(3\,x - 1.06) - .04(3.5\,x - 5) = 20.668.$

7. $\frac{.1\,x + .05}{2} - \frac{.01\,x + .005}{3} = .0009\frac{1}{3}.$

8. $\frac{.03(5-.4x)}{.2}+\frac{.7(2x-\frac{1}{2})}{.5}=9.914$..

9. A bank lent $500 in two parts, one at 6% and the other at 4%. The annual interest on the $500 was $25.90. What was each part?

10. In a certain solution of camphor and alcohol, the amount of camphor was 25% of the amount of alcohol. How much of each in 4.5 oz. of the mixture?

11. In a certain grade of concrete composed of pure cement and gravel the weight of the cement is $66\frac{2}{3}$% of the weight of the gravel. How many tons of pure cement are there in a block of this concrete weighing 6.6 T.?

237. The Linear Form $ax+b$. Every polynomial of the first degree can be put into the form $ax+b$. That is, by rearranging the terms suitably, it can be written as *the product of* x *by a number not involving* x, *plus an absolute term.* Hence, the form $ax+b$ is called a *general form* for all polynomials of the first degree in x.

Thus, $4x+3(2-5x)$ can be written $-11x+6$ which is in the form of $ax+b$, because -11 takes the place of a and $+6$ takes the place of b.

238. General Equations. Just as every linear polynomial is of the general form $ax+b$, so every equation of the first degree is equivalent to an equation of the form

$$ax+b=0;$$

consequently the latter is called *a general equation of the first degree with one unknown.*

239. General Solution. From the equation $ax+b=0$, (*1*)

we have $$ax=-b, \quad (2)$$

and hence, $$x=\frac{-b}{a}. \quad (3)$$

240. $\frac{-b}{a}$ is the *general form of the root* of the equation of the first degree. There is always *one root,* and only one.

241. To solve an equation of the first degree in an unknown:

Put the equation into the form $ax + b = 0$. *Then divide the absolute term,* b, *with its sign changed by the coefficient of* **x**.

WRITTEN EXERCISES

Simplify and solve according to Sec. 241:

1. $1 = 4x(3 + 7)$.
2. $6 + 8(1 - x) = 2$.
3. $5x + 3(x - 2) = 4x$.
4. $a = bx + c$.
5. $\frac{1}{x} = \frac{3}{4}$.
6. $\frac{x+1}{x+2} = 7$.
7. $\frac{x+1}{x} = \frac{4}{7}$.
8. $\frac{x}{1+x} = 2$.
9. $a = \frac{4y+5}{7-y}$.
10. $3px + 4bd = 7ax - b + d$
11. $w(1-a) + a(w-1) = 0$
12. $5z + 15 + \frac{3z+16}{5} = \frac{6z+91}{5}$.
13. $3(x + 1) + 7(x + 2) = 4(x + 3)$.
14. $(x - a)(x - b) = x^2 - a$.
15. $3bx - 7(x + b) + ac - cx = 0$.
16. $4 - \frac{5-x}{2} = \frac{3a+x}{4}$.
17. $2[3(x + 1) + 1] + 1 = 0$.
18. $\frac{1}{x+1} - \frac{a}{x+a} = 0$.
19. $(3 - x)(7 - x) = 10 + x^2$
20. $x(1 + a) + 2x(1 + b) = 3x(1 + c)$.
21. $(1 + 3x)^2 + (3 + 4x)^2 = (1 - 5x)^2$.
22. $(1 - x)(4 + x) + (2 + x)(6 + x) = 0$.
23. $(a - x)(b + x) = c^2 - x^2$.
24. $(x + b)(x + d) = (x + 1)^2$.
25. $\frac{1}{a+b} + \frac{a+b}{x} = \frac{1}{a-b} + \frac{a-b}{x}$.

Solve for a:

26. $ab + ac = 4.$ **27.** $ag + 3 = 2ah - 5.$

28. $(a - 1)(a + 2) = (a - b)(a + b).$

29. Solve this equation for e; also for b:

$$\frac{ab}{e} = bc + d + \frac{1}{e}.$$

Solve for g:

30. $v = at + \frac{gt^2}{2}.$ **31.** $gh + gh^2 = r^3.$

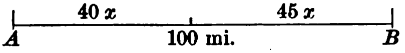
32. How much alcohol must be added to 2 qt. of a solution 90 % pure alcohol to make a solution 95 % pure?

Suggestion. Let x equal the number of quarts of alcohol to be added. Then, the new solution will contain $(2 + x)$ qt. The water in the new solution, or 5 % of $(2 + x)$ qt., will be the same as that in the original solution, or 10 % of 2 qt.

33. 18 % by weight of wheat is lost (as bran, etc.) in grinding it into flour. How many 60-lb. bushels of wheat are used in making 738 lb. of flour?

34. Divide the number n into two parts A, B, such that A shall be $\frac{3}{2}$ of B.

35. A train leaves A for B 100 mi. distant, traveling 40 mi. per hour. At the same time a train leaves B for A, traveling 45 mi. per hour. How far from A will they meet, and how long after starting?



Solution. 1. Let x be the number of hours from the time the trains start until they meet.

2. Then, the train from A travels $40x$ mi., and the train from B, $45x$ mi.

3. Therefore, $40x + 45x = 100$, as in the figure.

4. Therefore, $x = \frac{100}{85} = 1\frac{3}{17}$, the number of hours.

5. And $40x = 47\frac{1}{17}$, the number of miles from A.

Check. Find the distance from B and add to $47\frac{1}{17}$ mi.

36. A train leaves A for a station B, d miles distant; its rate is r miles per hour; h hours later another train leaves B for A, running R miles per hour. How far will each train have traveled when they meet?

SOLUTION. 1. Let $x =$ the number of hours traveled by the first train before they meet.

2. Then, $x - h =$ the number of hours traveled by the second train.
3. $rx =$ number of miles traveled by first train.
4. $R(x - h) =$ number of miles traveled by second train.
5. $\therefore rx + R(x - h) = d$, by the conditions of the problem.
6. $\therefore (r + R)x = d + Rh$, performing the operations.
7. $\therefore x = \dfrac{d + Rh}{r + R}$, solving (6).
8. $\therefore rx = \dfrac{r(d + Rh)}{r + R}$, the distance traveled by the first train.
9. And $R(x - h) = \dfrac{R(d - rh)}{r + R}$, the distance traveled by the second train.

37. Read the above problem, taking $d = 450$, $r = 50$, and $R = 40$, and $h = 4\frac{1}{2}$. Solve the problem by substituting these numbers in the expressions of steps 8 and 9.

242. Formula. An expression which shows how a desired number is to be obtained from given numbers is called a formula.

Thus, the expressions in steps 8 and 9 of Problem 36 are formulas.

A formula contains the solution of all problems which may be made by simply changing the numbers in the given problem.

WRITTEN EXERCISES

1. The sum of three numbers is 40. The second is 6 more than the first, and the third is the sum of the other two. Find the numbers.

2. The sum of three numbers is s; the second is a greater than the first, and the third is the sum of the other two. Find each number.

3. The sum of three numbers is s. The second is a less than the first, and the last is b less than the second. Find the numbers.

4. Apply Exercise 3 to the problem: The sum of three numbers is 35. The second is 3 less than the first, and the third is 4 less than the second. Find the numbers.

243. Special Problems. The solution of many problems is made easier by a proper selection of the unknown quantities.

WRITTEN EXERCISES

1. A certain number consists of two digits whose difference is 3; and, if the digits are interchanged, the number so formed is $\frac{4}{7}$ of the original number. Find the latter.

Suggestion. 1. Let x be the smaller digit.
2. Then $3 + x$ is the greater.
3. $\therefore 10(x + 3) + x$ is the given number.
4. $\therefore 10x + (x + 3)$ is the number with the digits interchanged.
5. According to the problem, $10x + (x + 3) = \frac{4}{7}[10(x + 3) + x]$.
6. Solve for x.

2. The tens' digit of a certain number exceeds the units' digit by 4, and when the number is divided by the sum of its digits, the quotient is 7. Find the number.

3. A boatman who rows $3\frac{1}{2}$ mi. in still water finds that it takes him 12 hr. to row upstream a distance that he can row down in 2 hr. What is the rate of the current?

Suggestion. Let x be the rate of the current. Then, the boatman rows upstream at the rate of $3\frac{1}{2} - x$ miles per hour, and downstream at the rate of $3\frac{1}{2} + x$ miles per hour.

4. The sum of three numbers, a, b, c, is 3036; a is the same multiple of 7 that b is of 4, and also the same multiple of 5 that c is of 2. What are the numbers?

Suggestion. Show that a, b, and c may be denoted by $7x$, $4x$, $\frac{7x}{5} \cdot 2$.

5. A number is composed of three digits each greater by one than that on its right; the difference between the number and $\frac{1}{4}$ of the number formed by reversing the order of the digits is 36 times the sum of the digits. Find the number.

Suggestion. A convenient selection of the digits is $x + 1$, x, $x - 1$.

REVIEW

WRITTEN EXERCISES

Solve for x:

1. $\frac{x+1}{x+2}+\frac{x+3}{x+4}=2.$
2. $(a-x)(b+x)=b-x^2.$
3. $(a+x)(b+x)=ab+x^2.$
4. $(a-x)(b-x)=b+x^2.$
5. $(a+bx)(a-bx)=(a^2+b^2x)(a-x).$
6. $(a+bx)(b-ax)=(a-bx)(ax-b).$
7. $(1-x)(2-x)-(3-x)(4-x)=5-x.$
8. $(a+x)(b-x)=a-x^2.$
9. $ax(bx+1)=bx(ax+1)+a+b.$
10. $(1+x)(2+x)+(5+6x)(1+8x)=(4+7x)^2$
11. $(a+x)(d+x)-(c-x)(a-x)=0.$

12. How much water must be added to 80 lb. of a 5 % solution of salt to obtain a 4 % solution?

13. Two men are 25 mi. apart and walk toward each other at the rates of $3\frac{1}{2}$ mi. and 4 mi. an hour respectively. After how long do they meet?

14. A man travels 50 mi. in an automobile in $3\frac{1}{4}$ hr. If his car runs at the rate of 20 mi. an hour in the country, and at the rate of 8 mi. an hour when within city limits, find how many miles of his journey are made in the country.

15. A train running 30 mi. an hour requires 21 min. longer to go a certain distance than does a train running 36 mi. per hour. What is this distance?

16. A and B set out on an automobile trip, A having $\frac{2}{3}$ as much money with him as B; after A had paid out $1 less than $\frac{2}{3}$ of his money, and B had paid $1 more than $\frac{7}{8}$ of his, it was found that B had left only half as much as A. How much had each at the outset?

17. In forming a regiment into a solid square 60 men were left over; but, when formed into a rectangle with 5 men more in front than before and 3 less in depth, there was one man wanting to complete it. Find the total number of soldiers in the regiment.

18. Two automobiles start from the same place; one goes east at the rate of 18 mi. an hour and the other west at 15 mi. an hour. In how many hours are they 330 mi. apart?

19. A train going from New York to Chicago at the average rate of 40 mi. an hour takes $4\frac{4}{5}$ hr. longer than one going 50 mi. an hour. Find the distance between these places.

20. Given $a =$ the amount, $r =$ the rate, and $p =$ the principal in a problem of simple interest, what is the time?

21. In Exercise 20 let $a = \$560$, $r = 4\%$, and $t = 3$ yr. Find the principal.

22. Divide the number a into two parts such that m times the greater may exceed n times the less by b.

23. A certain number consists of two digits whose difference is 5; and, if the digits are interchanged, the number so formed is $\frac{3}{8}$ of the original number. Find the latter number.

24. A certain number consists of two digits of which the tens' digit is 3 times the units' digit; and, if these digits are interchanged a number is formed which is less than the original number by 36. Find the original number.

25. Separate 275 into two parts such that $\frac{1}{5}$ of the smaller equals $\frac{1}{6}$ of the larger.

26. The sum of three consecutive even numbers is 306. Find the numbers.

27. One-fourth of a man's age now equals $\frac{1}{2}$ of what it was 20 yr. ago. Find his age now.

28. In a certain number of two digits the tens' digit exceeds the units' digit by 2; if the number is diminished by $\frac{3}{2}$ of the sum of its digits, they are interchanged. Find the number.

29. Separate 150 into two numbers such that if one be divided by 23 and the other by 27, the sum of the quotients is 6.

30. The current of a certain river is 6 mi. per hour. A steamer can go upstream in 9 hr., a distance that takes only 3 hr. to come down using the same power per hour. Find the rate of the steamer in still water.

31. A boatman who could row 5 mi. per hour in still water rowed a certain distance up a stream and back again. The current was 3 mi. per hour, and it took the boatman 10 hr. to make the round trip. How far up the stream did he go?

32. A can do a piece of work in 3 da. But A and B working together can do the same work in 2 da. How many days would it take B to do it alone?

SUGGESTION. Let $x =$ the number of days required by B. Then, A can do $\frac{1}{3}$ in 1 da. and B can do $\frac{1}{x}$ in 1 da. The sum of these multiplied by 2 will be the whole, or 1.

33. To a mass of metal composed of 4 parts of silver to 1 part of tin, enough silver is added to make a mass containing 6 parts of silver to 1 part of tin. How many ounces of silver are added per ounce of the original metal?

34. Twenty coins composed of dimes and quarters amount to $3.20. How many coins of each kind are there?

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. How is the degree of an equation determined? Sec. 225.

2. Define and illustrate *linear equation;* also *quadratic equation;* also *higher equation.* Secs. 226–228.

3. What is the *absolute term?* Sec. 230.

4. What is the first step in the solution of a fractional equation? Secs. 231–233.

5. How are decimal coefficients best removed? Sec. 236.

6. What is the *general form* of an equation of the first degree? What is the form of the root? Secs. 238–240.

7. What is a *formula?* Sec. 242.

CHAPTER XVII

RATIO, PROPORTION, AND VARIATION

RATIO

244. Ratio. The quotient of two numbers of the same kind is often called their **ratio.**

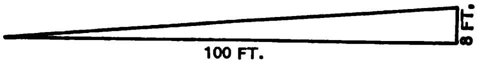
The following examples illustrate the use of the word "ratio":

A solution consists of sulphuric acid and water in the ratio of 2 to 3. This means that $\frac{2}{5}$ of the whole is sulphuric acid and $\frac{3}{5}$ is water.

Sterling silver requires a little other metal (alloy) to harden the silver in it; the ratio by weight of the amount of pure silver to the entire mass is usually 925 to 1000. This means that $\frac{925}{1000}$ of the whole is silver.

The specific gravity of a solid is the ratio of its weight to the weight of an equal volume of water under standard conditions.

The birth rate per annum in a city is said to be 23 when the ratio of the total number of births in a certain year to the total number of inhabitants at the beginning of that year is that of 23 to 1000.



A road bed is said to have an 8% grade when the ratio of the vertical rise to the horizontal distance is that of 8 to 100.

245. Ratio is commonly expressed by a fraction.

Thus, the ratio of 2 to 3 is written $\frac{2}{3}$; the old form 2 : 3 is less convenient in calculation.

Before ratios were written in fractional form it was convenient to have names for the terms. Thus, the first number was called the *antecedent*, and the second number the *consequent*.

In division the divisor may be abstract or concrete. If it is abstract, the quotient is of the same character as the dividend. If it is concrete, the dividend must be concrete and expressed in the same unit.

Consequently, we cannot speak directly of the ratio of 12 gal. to 3 qt.; we must first express both numbers in the same unit, as 48 qt. to 3 qt.

Similarly, when we speak of the ratio of the distance to the time, we mean the ratio of the corresponding abstract numbers, as in Sec. 244.

WRITTEN EXERCISES

1. How much pure silver is there in 200 oz. of sterling silver (Sec. 244)?

SUGGESTION. $\dfrac{x}{200}=\dfrac{925}{1000}$.

2. A silversmith buys $46\frac{1}{4}$ oz. of pure silver; how much sterling silver can be made from it?

SUGGESTION. The equation is $\dfrac{46\frac{1}{4}}{x}=\dfrac{925}{1000}$.

3. What is the rate per second of a train which travels uniformly 635 ft. in 5 sec.?

4. The weight of a piece of gold is 94.5 oz., and the weight of an equal volume of water is 5 oz. What is the specific gravity (Sec. 244) of the gold?

5. There are 4053 births in a certain city, making its birth rate 21 (Sec. 244). What is the population of the city?

6. If the population of a city is p and the birth rate is b, indicate the number of births.

7. The top of a mountain pass is 1200 ft. vertically above the level of the base; the top is reached by a zigzag road 5 mi. long. What is the average grade of the road?

8. A road m mi. long, measured horizontally, ascends to a height of f ft. above the level of its starting point. Indicate the average grade of the road.

9. Two men, A and B, divide \$963 of profits so that A's part is to B's in the ratio of 2 to 1. How many dollars has each?

SUGGESTION.
1. Let x be the amount A receives.
2. Then, $963-x$ is the amount B receives.
3. $\therefore \dfrac{x}{963-x}=\dfrac{2}{1}$, the ratio of the shares, as given.
4. Clear the equation of fractions and solve for x.

10. Two partners, A and B, divide \$575 in the ratio of 2 to 3. How many dollars does each receive?

246. Special Notations. When a letter stands for different values of the same variable quantity, these values are commonly designated either by:

1. *Small letters and capital letters.*

For example, if we have to represent the rates of two moving bodies in the same problem, we may use r for the rate of one and R for the rate of the other.

2. *Primes and seconds.*

For example, if two weights occur in the same problem, we may express one by w' and the other by w'', read: "w prime" and "w second."

3. *Subscripts.*

If a problem has three quantities indicating time, we may express these by t_1, t_2, t_3, read: "t sub-one," "t sub-two," "t sub-three," or simply "t-one," "t-two," and "t-three."

Primes and subscripts should be carefully distinguished from exponents or coefficients, because they have no significance other than to distinguish numbers of the same kind one from another.

WRITTEN EXERCISES

1. Solve for r the equation $\frac{r}{3}=\frac{5}{R}$.

2. Solve the equation in Exercise 1 for R.

3. Solve for r' the equation $\frac{1}{r'}=\frac{r''}{5}$.

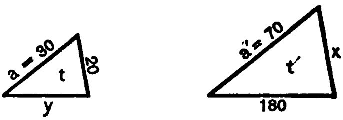
4. Solve the equation in Exercise 3 for r''.

5. Solve $\frac{t_1}{.5}=\frac{.1}{t_2}$ for t_1; also for t_2.

6. Solve $3t_1=\frac{5}{6}tt_2$ for t_2. Solve the equation for t.

7. It is known that in triangles whose corresponding angles are equal, the corresponding sides have the same ratio.

In the figure, what is the ratio of a to a'? Of 20 to x? Find x. What is the ratio of y to 180? Find y.



Supply the numbers to fill the blanks in the table:

	Substance	w = weight of substance	w' = weight of an equal volume of water	Specific gravity = $\frac{w}{w}$
8.	Lead	56.5 oz.	5 oz.	——
9.	Oak	12.75 oz.	15 oz.	——
10.	Tin	153.3 lb.	21 lb.	——
11.	Coal	18 T.	10 T.	——
12.	Alcohol	13.6 oz.	17 oz.	——

247. A property of ratio:

The numerical value of a ratio whose numerator is less than its denominator is increased by adding the same positive number to both terms.

The numerical value of a ratio whose numerator is greater than its denominator is decreased by adding the same positive number to both terms.

1. For, let $\frac{a}{b}$ be the given ratio and x the given positive number.

2. To determine whether $\frac{a}{b}$ or $\frac{a+x}{b+x}$ is the greater, subtract the second ratio from the first.

3. Then, $\frac{a}{b}-\frac{a+x}{b+x}=\frac{ab+ax-ab-bx}{b(b+x)}=\frac{(a-b)x}{b(b+x)}$.

(1) If $a < b$, the last ratio is a negative number, therefore the ratio of less inequality $\frac{a}{b}$ would be increased by adding x to each of its terms.

(2) If $a > b$, the last ratio is a positive number, therefore the ratio of greater inequality $\frac{a}{b}$ would be diminished by adding x to each of its terms.

A ratio whose numerator is less than its denominator was formerly called a **ratio of less inequality.**

A ratio whose numerator is greater than its denominator was formerly called a **ratio of greater inequality.**

(3) If $a = b$, the last ratio is zero, therefore the ratio $\frac{a}{b}$, if equal to unity, would be unchanged by adding x to each of its terms.

A knowledge of this property will remove the temptation to say that adding or subtracting the same number from both terms of a fraction does not alter its value.

ORAL EXERCISES

1. What is the effect on $\frac{2}{3}$ of adding 4 to each term?
2. What is the effect on $\frac{3}{2}$ of adding 5 to each term?
3. What is the effect on $\frac{4}{5}$ of adding a to each term?
4. What is the effect on $\frac{5}{4}$ of adding b to each term?
5. What is the effect on $\frac{6}{6}$ of adding c to each term?
6. Which is the greater, $\frac{2}{3}$ or $\frac{4}{5}$? Also $\frac{4}{7}$ or $\frac{2}{5}$? Why?
7. Which is the greater, $\frac{a}{b}$ or $\frac{a+1}{b+1}$? Why?
8. Which is the less, $\frac{a}{b}$ or $\frac{a-2}{b-2}$? Why?
9. Which is the less, $\frac{a}{b}$ or $\frac{2a}{a+b}$? Why?
10. Compare $\frac{n}{n+1}$ and $\frac{n+1}{n+2}$; also, $\frac{n-2}{n+1}$ and $\frac{n}{n+3}$.

PROPORTION

248. Proportion. An equation between two ratios is called a **proportion.**

Thus, $\frac{2}{3}=\frac{6}{9}$, $\frac{3a}{7a}=\frac{6b}{14b}$, $\frac{25}{5}=\frac{5}{1}$, $\frac{5}{6}=\frac{3}{x}$ are proportions.

A proportion is usually read in one of two ways: For example, $\frac{3}{5}=\frac{a}{x}$ is read "3 is to 5 as a is to x," or "the ratio three-fifths equals the ratio a over x."

249. The numbers forming one of the ratios are said to be 'proportional to" the numbers forming the other.

Thus, in the proportion $\frac{12}{9}=\frac{20}{15}$ the numbers 12 and 9 are proportional to 20 and 15.

250. The terms "proportional" and "proportionally" are used with the meaning "in the same ratio."

For example:

The express rate from Chicago to New York is $2.50 per 100 lb., the excess above 100 lb. being charged proportionally. This means the charge for the excess has the same ratio to the excess as 2.50 has to 100.

Of two men in business one furnished $\frac{2}{3}$ of the capital, and the other $\frac{1}{3}$ of it. They divided their gain of $9000 in proportion to their capitals. This means that they divided the $9000 into two parts having the ratio of 2 to 1.

251. The expression *pro rata* is often used with the same meaning as "proportionally" or "in the same ratio."

For example:

A and B hire an automobile for a trip and agree to pay $\frac{3}{5}$ and $\frac{2}{5}$ of the rental respectively, and other expenses occurring on the trip *pro rata*. This means that the other expenses are to be divided between A and B in the same ratio as the rental of the automobile.

WRITTEN EXERCISES

1. Two families of 3 members and 5 members respectively camp out together at an expense of $160; they divide this amount in proportion to the size of the families. How much did each family pay?

2. A man hires a piano at $ 5 per month of 30 days, and is to pay *pro rata* for any part of a month. What does he pay, if he keeps the piano 51 days? d days?

3. Three farmers share in the purchase of a steam thresher. A pays $ 400, B pays $ 600, and C pays $ 1000. In the course of the year the thresher is rented to other farmers 27 days at $ 10 per day, and the earnings divided among the owners *pro rata*. What does each receive?

4. If the thresher of Exercise 3 is rented d days at r dollars per day and the earnings divided *pro rata*, what does each owner receive?

5. It cost R dollars to repair the thresher, and the cost is divided *pro rata* among the owners. What does each pay?

WRITTEN EXERCISES

1. Find x in the proportion $\frac{x}{15} = \frac{160}{25}$. *(1)*

SOLUTION:

Multiplying both members by 15, $x = \frac{15 \times 160}{25}$. *(2)*

Simplifying the fraction in (2), $x = 96$. *(3)*

2. Find x in the proportion $\frac{7}{x} = \frac{.14}{42}$. *(1)*

SOLUTION:

Clearing of fractions, $7 \times 42 = .14\,x$. *(2)*

Simplifying (2), $x = \frac{7 \times 42}{.14} = 2100$. *(3)*

3. Solve the proportion $\frac{h}{l} = \frac{h'}{l'}$ for h. For h'. For l'.

4. Solve the proportion $\frac{p}{P} = \frac{W}{w}$ for p. For P. For W. For w.

5. Solve the equation $\frac{p}{p'} = \frac{b}{b'}$ for p. For b. For b'. For p'.

6. In Exercise 5 what is the value of p', if $p = 15$, $b = 28$, and $b' = 30.5$?

7. Two partners, A and B, in business divide $9000 between them in the ratio of 2 to 1. How much does each receive?

8. At $2.80 for 8 hours' work, overtime paid proportionally, how much does a workman receive for $2\frac{1}{2}$ hr. overtime?

9. After rents rose $\frac{1}{3}$ and other things in proportion, a family's expenses for one month were $132. What were their expenses before the rise?

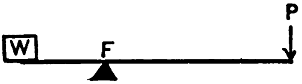
10. 1 cu. ft. of lime and 2 cu. ft. of sand are used in making 2.4 cu. ft. of mortar. How much of each is needed to make 72 cu. ft. of mortar?

11. In making glass, 1, 5, and 15 portions by weight of chalk, potash, and sand respectively are used. How many pounds of each are required to produce 1176 lb. of glass?

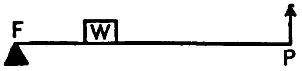
252. Problems of the Lever:

1. The point of support of a lever is called the fulcrum and in the figure is denoted by F.

If P denotes the power and W the weight (expressed in the same unit), and if p denotes the length of the arm FP and w the length of the arm FW (both expressed in the same unit), then it is known that



$$\frac{w}{p}=\frac{P}{W}$$



In words: *The ratio of the lengths of the arms equals the reciprocal of the ratio of the corresponding forces.*

Express w in terms of P, W, and p. Express W in terms of P, p, and w.

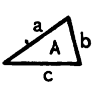
2. Find p if $P = 10$ lb., W $= 60$ lb., $w = 2$ ft.

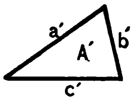
Supply the numbers to fill the blanks in the following table concerning levers:

	p	w	P	W
3.	( )	20 in.	1.60 lb.	90 lb.
4.	1.5 ft.	( )	2.25 lb.	1125 lb.
5.	12 ft.	$\frac{1}{2}$ ft.	( )	960 lb.
6.	.3 ft.	12.9 ft.	2.5 T.	( )

253. Problems of Similar Triangles:

1. Areas of similar triangles (those of the same shape) are proportional to the squares of the lengths of their corresponding sides. Let A and A' be the areas of two similar triangles, and a and a' be a pair of corresponding sides. Then $\frac{A}{A'}=\frac{a^2}{a'^2}$.





Solve the proportion $\frac{A}{A'}=\frac{a^2}{a'^2}$ for A. For A'. For a. For a'.

2. Express the ratio of a to a' in Exercise 1.

3. The areas of two similar triangles are 64 sq. in and 625 sq. in. What is the ratio of any pair of corresponding sides?

4. If the side a of the smaller triangle in Exercise 3 is 8 in., what is the corresponding side a', in the larger triangle? If the side b of the smaller triangle is 5 in., what is b'?

5. The lengths of a pair of corresponding sides in two similar triangles are 3 ft. and 8 ft.; the area of the larger one is 640 sq. ft. What is the area of the smaller one?

VARIATION

254. If related numbers change so as always to remain in the same ratio, one is said to *vary as* the other.

For example:

At 5¢ per pound, the amount paid varies as the number of pounds purchased.

If a body moves at the same rate, the distance varies as the time of motion.

If $100 is placed at simple interest, the amount of interest varies as the time.

255. "Varies as" is thus seen to be merely another expression for "is proportional to" or "varies proportionally with."

ORAL EXERCISES

1. How much more can a man lift with a lever 8 ft. long than with a lever 4 ft. long? What is the weight lifted proportional to? What does the weight vary as?

2. How much more does $10 earn at 6 % simple interest in one year than $5? When the rate and time are the same what does the interest vary as?

3. When the principal and the rate are the same, what does the simple interest vary as?

4. An automobile is running at a uniform rate. The distance traveled varies as what number?

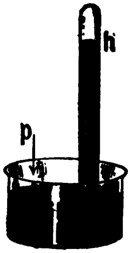
5. If the machine in Exercise 4 traveled 90 mi. in 3 hr., how far would it travel in 5 hr.?

WRITTEN EXERCISES

1. When the grade of a roadbed is $8\frac{1}{2}\%$, what is the rise in a horizontal distance of 175 ft.?

2. What weight can a man weighing 175 lb. raise with a lever 5 ft. long, if the fulcrum is 6 in. from the end acting on the weight?

3. What weight can he raise, if he weighs 180 lb., the lever being $4a$ ft. long, and the support being a ft. from the end acting on the weight?



4. The height to which the mercury rises in a barometer varies as the pressure of the air on the mercury. Let the heights at two readings be h and h', and the corresponding pressure p and p'; then $\frac{p}{p'} = \frac{h}{h'}$. Express p in terms of h, h', and p'.

5. In Exercise 4, let two readings of the barometer be $h = 28.5$ and $h' = 29.5$, and the corresponding pressures be p and $p' = 15$ lb. What is the value of p?

Supply the blanks in the following table of readings:

	p	p'	h	h'
(6)	( ) lb.	14.5 lb.	30 in.	29 in.
(7)	14.75 lb.	( ).	29.5 in.	30 in.
(8)	15 lb.	15.25 lb.	( )	30.5 in.
(9)	14.2 lb.	14.8 lb.	28.75 in.	( )

PROPERTIES OF PROPORTION

256. Means and Extremes. The middle numbers in a proportion are called the **means** and the end numbers the **extremes.**

Thus, in the older form of writing a proportion, $a : b :: c : d$, b and c are called the *means* and a and d the *extremes.* When so written, the proportion is read "a is to b as c is to d." At present, it is more customary to use the form $\frac{a}{b} = \frac{c}{d}$ and to read it "a over b equals c over d."

257. Fourth Proportional. In the proportion $\frac{a}{b}=\frac{c}{d}$, the fourth number, d, is called the **fourth proportional.**

Numbers that are said to form a proportion must be placed in the proportion in the order in which they are given. Thus, if 3, 5, 10, and x are said to form a proportion, then $3:5=10:x$.

258. Third Proportional. In the proportion $\frac{a}{b}=\frac{b}{c}$, the third number, c, although in the fourth place, is called the **third proportional** to a and b.

259. Mean Proportional. In the proportion $\frac{a}{b}=\frac{b}{c}$, b is called the **mean proportional** between a and c.

260. Relation to the Equation. A proportion is an equation expressing the equality of two ratios.

EXAMPLES

1. Find the fourth proportional to the numbers 6, 8, 30.

Let x be the fourth proportional, then, by Sec. 257, $\frac{6}{8}=\frac{30}{x}$. *(1)*

Multiplying by $8x$, $6x=8\cdot 30$. *(2)*

Dividing by 6, $x=40$. *(3)*

Therefore 40 is the fourth proportional to 6, 8, 30.

2. Find the third proportional to 5 and 17.

Let x be the third proportional, then, by Sec. 258, $\frac{5}{17}=\frac{17}{x}$. *(1)*

Multiplying both members by $17x$, $5x=17^2$. *(2)*

Solving *(2)*, $x=\frac{289}{5}=57\frac{4}{5}$. *(3)*

Therefore $57\frac{4}{5}$ is the third proportional to 5 and 17.

WRITTEN EXERCISES

1. Write m, n, and p so that p shall be the third proportional to m and n.

2. Write m, n, and p so that n shall be the mean proportional between m and p.

3. Solve: $-\frac{5}{x}=\frac{1}{4}$.

4. Solve: $\frac{1.21}{x}=\frac{x}{.09}$.

Solve:

5. $\frac{x}{11}=\frac{7}{1331}$.

6. $x \div 16 = \frac{1}{4} \div x$.

7. $-24 \div x = 2x \div -12$.

8. $\frac{1}{6}=\frac{42}{x}$.

9. $\frac{75}{-x}=\frac{-x}{3}$.

10. Find the third proportional to 8 and 5.

11. Find the third proportional to -6 and -4.

12. Find the mean proportionals between the following numbers: 9 and 16; -25 and -4.

13. If a sum of money earns $48 interest in 5 yr., how much will it earn in 16 yr. at the same rate per cent?

14. A city whose population was 40,000 had 2500 school children; the total population increased to 48,000, and the number of children of school age increased proportionally. How many children of school age were there then?

15. What number must be added to each of the four numbers, 5, 29, 10, 44, to make the results proportional?

261. Relation of Means to Extremes. *In any proportion the product of the means equals the product of the extremes,*

For, $\frac{a}{b}=\frac{c}{d}$, and multiplying both members by bd, $ad = bc$.

262. Conversely, *If the product of two numbers equals the product of two other numbers, the four numbers can be arranged in a proportion, the two factors of one product being the means, and the two factors of the other product the extremes.*

For, if $ad = bc$, divide both members by bd, and $\frac{a}{b}=\frac{c}{d}$.

Also, if

$$a^2 - b^2 = xy,$$

$$\frac{a+b}{x}=\frac{y}{a-b}.$$

263. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$.

For, if two numbers are equal, their reciprocals are equal. **Sec. 219.**

The older form of statement still sometimes used is: If four numbers form a proportion, they are in proportion by **inversion**.

264. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$.

For, multipiying both members by $\frac{b}{c}$, $\frac{b}{c}\cdot\frac{a}{b}=\frac{b}{c}\cdot\frac{c}{d}$,

and canceling b and c, $\frac{a}{c}=\frac{b}{d}$.

The older form of statement is: If four numbers form a proportion they are in proportion by **alternation.**

265. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$.

Also $\frac{a-b}{b}=\frac{c-d}{d}$.

For, adding $+1$ to both members, $\frac{a}{b}+1=\frac{c}{d}+1$.

Therefore, performing the processes, $\frac{a+b}{b}=\frac{c+d}{d}$.

Similarly, by adding -1 to both members of the given proportion the second form results.

The older form of statement is: If four numbers form a proportion, they are in proportion by **composition** or by **division.**

266. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

For, from Sec. 265, $\frac{a+b}{b}=\frac{c+d}{d}$,

and $\frac{a-b}{b}=\frac{c-d}{d}$.

Dividing these equations, member for member,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

The older form of statement for this is: If four numbers form a proportion, they are in proportion by **composition and division.**

267. The **mean proportional** between two numbers is the square root of their product.

For, from $\frac{a}{b}=\frac{b}{c}$, Sec. 259, we find $b=\sqrt{ac}$.

268. Series of Equal Ratios. The following theorem concerning ratios is sometimes used:

In a series of equal ratios (a continued proportion), the sum of the numerators divided by the sum of the denominators is equal to any one of the ratios.

PROOF: Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots$ be the equal ratios and r be their common value. (*1*)

Then $\frac{a}{b}=r$, $\frac{c}{d}=r$, $\frac{e}{f}=r$, and so on. (*2*)

$\therefore$ $a=br$, $c=dr$, $e=fr$, and so on. (*3*)

$\therefore$ adding the equations in (*3*),

$$(a+c+e\cdots)=(b+d+f\cdots)r. \qquad (4)$$

$\therefore$ solving (*4*) for r, $\frac{a+c+e\cdots}{b+d+f\cdots}=r.$ (*5*)

$\therefore$ $\frac{a+c+e\cdots}{b+d+f\cdots}=\frac{\text{the sum of the numerators}}{\text{the sum of the denominators}}=\frac{a}{b}$, or $\frac{c}{d}$, and so on, since r is any of these ratios. (*6*)

269. These properties may be applied to certain equations.

EXAMPLES

1. Given $\frac{a}{b}=\frac{c}{d}$; show that $\frac{3a-5b}{5b}=\frac{3c-5d}{5d}$. (*1*)

Multiplying both members of the given equation by 3, $\frac{3a}{b}=\frac{3c}{d}$. (*2*)

Dividing (2) by -5, $\frac{3a}{-5b}=\frac{3c}{-5d}$. (*3*)

Applying Sec. 265 to (3), $\frac{3a-5b}{5b}=\frac{3c-5d}{5d}$. (*4*)

2. Three numbers are in the ratio of 3 to 4 to 7 and their sum is 42. Find the numbers.

SOLUTION:

1. Let $3x$ be the first number, then $4x$ and $7x$ are the others.
2. $\therefore$ $3x+4x+7x=42$, by the conditions of the problem.
3. $\therefore$ $14x=42$ and $x=3$.
4. The numbers are 9, 12, and 21 by steps (*1*) and (*3*).

3. Solve:
$$\frac{2x-2}{4-9x}=\frac{2}{9x}. \tag{1}$$

Applying Sec. 268,
$$\frac{2x}{4}=\frac{2}{9x}. \tag{2}$$

Simplifying (2),
$$\frac{x}{2}=\frac{2}{9x}. \tag{3}$$

Solving (3),
$$9x^2=4, \tag{4}$$

and
$$x=\pm\tfrac{2}{3}.$$

Test by substitution.

WRITTEN EXERCISES

1. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a-4b}{4b}=\frac{c-4d}{4d}$.

2. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{b}{b-a}=\frac{d}{d-c}$.

3. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{2a}{a-b}=\frac{2c}{c-d}$.

4. Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a+c-e}{b+d-f}=\frac{c}{d}$.

5. Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a^2+c^2+e^2}{b^2+d^2+f^2}=\frac{e^2}{f^2}$.

6. Given $\frac{m}{n}=\frac{p}{q}=\frac{r}{s}$, show that $\frac{2m-3p+5r}{2n-3q+5s}=\frac{m}{n}$.

7. Given $\frac{x}{a}=\frac{b}{c}=\frac{e}{d}$, show that $x=\frac{a(b+e)}{c+d}$.

8. Write in the form of a proportion (Sec. 262):

(1) $a^2+2ab+b^2=cd$.
(2) $a^2-b^2=a(b+c)$.
(3) $x^2-5x+6=7x$.

9. Find x in $\frac{x-7}{x+2}=\frac{5x+7}{3x-2}$.

10. There are two numbers with the ratio of 3 to 5 and one is 24 less than the other. Find the numbers.

REVIEW

ORAL EXERCISES

1. What is the fourth proportional in $\frac{3}{3}=\frac{5}{d}$?

2. What is the third proportional in $\frac{3}{5}=\frac{5}{x}$?

3. What is the mean proportional between 4 and 9?

4. State the principle according to which it follows from

$$\frac{a}{b}=\frac{c}{d} \text{ that } \frac{a+b}{b}=\frac{c+d}{d}.$$

5. What is the value of x in $\frac{x}{a}=\frac{b}{x}$?

WRITTEN EXERCISES

Find the fourth proportional to:

1. x, y, ab. 2. p^2, q, m^2q. 3. x^2, y^3, z^3.

Find the mean proportional between:

4. a^2, a^2b^2. 5. $16, 4a^2b^2$. 6. $27x^2y^3, 3y$.

When $\frac{a}{b}=\frac{c}{d}$ show that:

7. $\frac{ac}{bd}=\frac{a^2}{b^2}$.

8. $\frac{a}{c}=\frac{a-b}{c-d}$.

9. $\frac{a+2b}{2b}=\frac{c+2d}{2d}$.

10. $\frac{2a-3b}{3b}=\frac{2c-3d}{3d}$.

11. $\frac{4a-5b}{5b}=\frac{4c-5d}{5d}$.

12. $\frac{a-b}{c-d}=\frac{b}{d}$.

13. The ratio of two numbers is 3 to 5 and their sum is 48. Find the numbers.

14. The lengths of the sides of a triangle are in the ratio of 3 to 4 to 5 and their sum is 120 ft. What is the length of each side?

15. Denoting by b the arm of a lever on which the power p is applied and by a the arm on which the weight w is applied:

What force is required to raise 500 lb., if $a = 2$ and $b = 10$? Also, if $a = 8$ and $b = 40$? Also, if $a = 9$ and $b = 36$?

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define *ratio*; also *proportion*. Secs. 244, 248.

2. What is the effect of adding the same positive number to both terms of a fraction? Sec. 247.

3. When are two numbers proportional to two other numbers? Sec. 249.

4. Explain "*proportional,*" "*pro rata,*" and "*in the same ratio.*" Secs. 250, 251.

5. What is meant by "*one quantity varies as another*"? Sec. 254.

6. What is the relation of variation to proportion? Sec. 255.

7. Define the *extremes* and *means* of a proportion. Sec. 256.

8. What is the *fourth proportional* to three numbers? Sec. 257.

9. What is the *third proportional* to two numbers? Sec. 258.

10. What is the *mean proportional* between two numbers? Sec. 259.

11. When can four numbers form a proportion? Sec. 262.

12. Illustrate the result of taking a proportion by inversion; also by alternation; by composition; by division; by composition and division. Secs. 263–266.

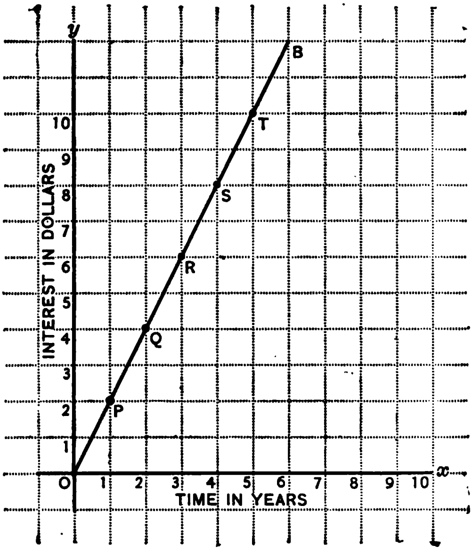
13. State a property of a series of equal ratios. Sec. 268.

CHAPTER XVIII

GRAPHS OF LINEAR EQUATIONS

270. **Preparatory:**

1. If $100 is lent at 2 % simple interest, show graphically how the amount of interest (I) varies with the number of years (t).



In the diagram the spaces on the horizontal scale represent the number of years, and those on the vertical scale the number of dollars. The positions of the points on the line OB show the amounts of interest for the periods of time indicated on the horizontal scale. Thus, point P shows that the interest is $2 when the time is 1 yr.; point Q shows that the interest is $4 when the time is 2 yr.

2. What does point R show? Point O? Point S? Point T?

3. What does the point halfway between Q and R show?

The number of dollars interest is always twice the number of years. This is expressed by the formula, $I = 2t$.

From the graph we can read either the amount of interest for a given time, or the time required to earn a given interest.

ORAL EXERCISES

From the graph read:

1. The amount of interest on $100 at 2% for 2 yr. For 3 yr. For $1\frac{1}{2}$ yr.

2. The time in which $100 at 2% will earn $4. $5. $3.

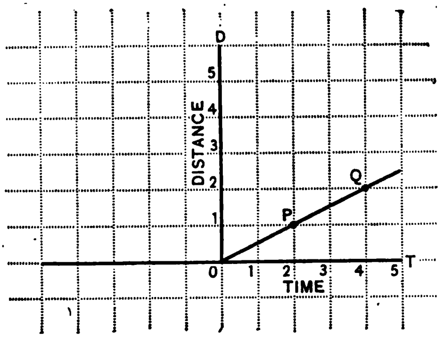
WRITTEN EXERCISES

1. Make a graph like that on page 198, taking the rate of interest to be 3%.

2. From the graph answer questions like 1 and 2 above. State the relation between interest and time in this case. Write an equation expressing this relation.

3. Treat similarly each of the rates: 4%; $2\frac{1}{2}$%; 5%; 6%.

4. In the figure values of t are represented on the horizontal line OT, and corresponding values of d on the line OD. What is the value of t for point P? Of d for point P? What is the value of t for point Q? Of d for point Q? Every value of d is what part of the corresponding value of t? Express this relation by an equation.



5. An elevator goes up at the rate of 4 ft. per second. What distance does it ascend in 2 sec.? In 3 sec.? In 5 sec.? In 1 min.?

6. Letting $d =$ the number of feet passed over in any number of seconds (t), and using horizontal spaces to represent the number of seconds, and vertical spaces to represent the number of feet, make a graph to represent the relation $d = 4t$.

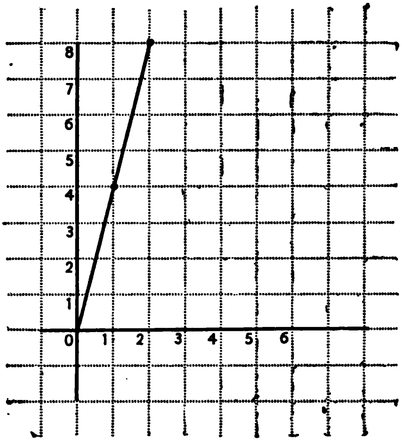
271. Before constructing a graph, the corresponding values of the letters may be conveniently arranged in a table as in the following example:

Construct the graph of $y = 4x$.

TABLE

x	y
0	0
1	4
$1\frac{1}{2}$	6
2	8
$2\frac{1}{2}$	10
3	12
4	16
5	20

GRAPH



WRITTEN EXERCISES

Construct the graphs of:

1. $2y = 3x$.
2. $4y = x$.
3. $y = \frac{1}{3}x$.
4. $y = \frac{1}{2}x$.
5. $y = \frac{2}{3}x$.
6. $\frac{1}{2}y = x$.
7. $y = \frac{3}{2}x$.
8. $y = \frac{2}{3}x$.
9. $d = \frac{1}{4}t$.
10. $d = \frac{3}{4}t$.
11. $d = 5t$.
12. $\frac{1}{3}d = t$.
13. $d = 3\frac{1}{2}t$.
14. $2\frac{1}{2}y = x$.
15. $1\frac{1}{4}x = y$.

272. Graphs may be constructed for negative values as well as for positive values of the numbers involved.

EXAMPLES

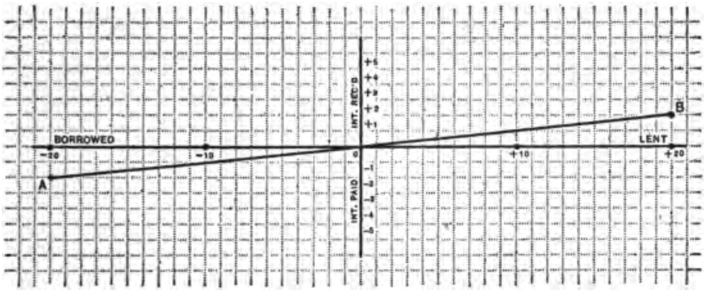
1. If we regard money borrowed (which is the opposite to money lent) as negative, and interest paid (which is the opposite to interest received) as negative, we may express by a single line the changes in interest and principal, both for money borrowed and for money lent.

The table shows the change in interest at 5% for 2 yr. corresponding to the change in the principal from + $20 to − $20, the positive values denoting money lent and interest received, the negative ones denoting money borrowed and interest paid.

TABLE

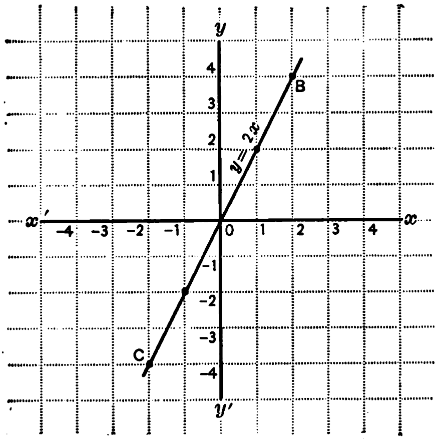
$p =$	20	10	0	−10	−20
$i =$	2	1	0	−1	−2

The line AOB is the graph representing these changes.



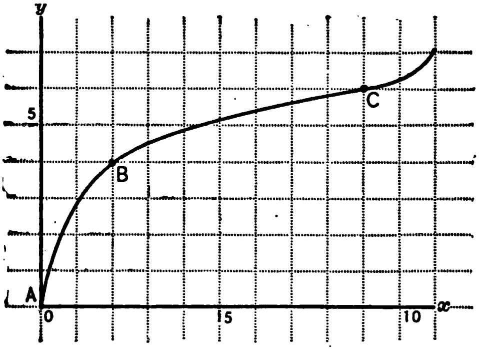
2. The line CB in the figure below is the graph of the equation $y = 2x$, for both positive and negative values. The line BO is the graph for all positive values of x and y and the line OC, the extension of OB, is the graph for negative values. The negative values of x are marked off to the left of O and the negative values of y downward from O.

y	x
−4	−2
−2	−1
−1	$-\frac{1}{2}$
0	0
2	1
3	$1\frac{1}{2}$
4	2



273. Any change that takes place throughout at a constant rate is said to take place "uniformly."

For example, a body moving continually at the same speed is said to move uniformly. That *the graph of uniform change* will be a straight line appears from the fact that the straight line is the only path along which a point moves upward or downward uniformly. Along any curved line it moves upward or downward more rapidly at some times than at others. Thus, in the figure the increase from A to B is much more rapid than from B to C.



If x and y vary, subject to the equation $y = 4x + 3$, or to $y = mx + b$, the change will be uniform. For, in the first case, y changes by 4 times the change in x, and, in the second case, by m times the change in x; that is, y varies uniformly with x. Since we know that only straight lines represent uniform change, we know that the graphs of the above equations are straight lines.

ORAL EXERCISES

1. How many points are necessary to fix the position of a straight line?

2. How many points must be fixed in order to draw a graph which is known to be a straight line?

3. When several points are constructed corresponding to the values of the unknowns in an equation and a straight line is drawn through any two of them, where will the others lie?

274. Constructing the graph of an equation is called **plotting** the equation.

WRITTEN EXERCISES

Plot the following equations:

1. $3y = x$.
2. $y = 3x$.
3. $c = \frac{1}{2}p$.
4. $y = \frac{1}{3}x$.
5. $p = 4t$.
6. $s = \frac{1}{4}t$.
7. $2x - y = 0$.
8. $5p - w = 0$.
9. $x - 6y = 0$.
10. $y = 2x + 1$.
11. $y = 1 - 2x$.
12. $y = x - 4$.
13. $y = x + 4$.
14. $y = 5x$.
15. $y = -3x$.
16. $y = x$.
17. $y = \frac{1}{2}x + 6$.
18. $y = .3x - 2$.

275. Preparatory.

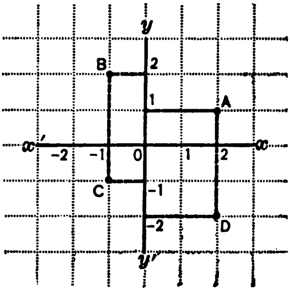
1. In the diagram how many spaces is point A to the right of line yy'? How many spaces is point A above the line xx'?

The position of point A is fixed by the distances 2, 1. (It is customary to name the distance along the line xx' first.)

2. How many spaces is point B to the left of line yy'? How many spaces above xx'?

The position of point B is fixed by the distances -1, 2.

3. How far is point C to the left of yy'? How far below xx'?



The position of point C is fixed by the distances -1, -1.

4. How far is point D from yy'? From xx'? What distances fix the position of this point?

The position of point D is fixed by the distances $+2$, -2.

276. Axes. The lines of reference designated by xx' and yy' are called the **axes**.

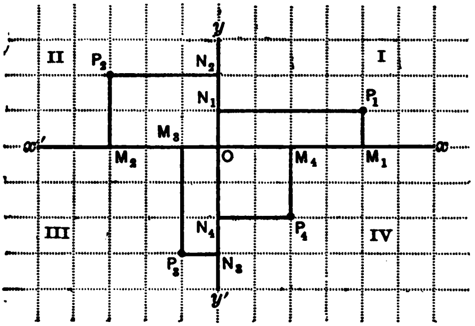
277. Quadrants. The axes divide the diagram into four quarters called *quadrants.* These are numbered I, II, III, and IV as shown in the figure in Section 280.

278. The position of the point P_1 is fixed by its perpendicular distances, P_1M_1, P_1N_1, from the axes. These two perpendicular distances are called the **coördinates** of point P_1. Similarly, the coördinates of P_2 are P_2M_2, P_2N_2.

The position of any point is fixed by its two coördinates.

279. The point of intersection of the axes is called the **origin of coördinates.**

280. Abscissas and Ordinates. The distance of a point from the axis yy', as P_1M_1, is called the **abscissa** of the point, and the distance of the point from the axis xx' is called the **ordinate** of the point.



The ordinates of all points in quadrants I and II are positive, and of those in III and IV are negative.

The abscissas of all points in quadrants I and IV are positive, and those of all points in II and III are negative.

281. Variables. A number symbol that may assume different values is called a **variable.**

Thus, the number of degrees of temperature from time to time, the number of hours from sunrise to sunset, the price of wheat, the number of inhabitants of the United States, are variable numbers. They measure physical or other quantities that are in themselves variable.

282. Constants. A number that has a fixed value is called a **constant.**

Thus, 3, $\sqrt{5}$, -4, are constants.

283. When numbers are indicated by letters, the conditions of the problem must specify which are constant and which are variable.

It is customary to use the earlier letters of the alphabet to denote constants, and the later ones to denote variables, but it is not necessary to do so.

284. Function. When the value of one variable depends upon that of another the first variable is said to be a **function** of the second.

The function is called the **dependent** variable, and the other the **independent variable.**

For example:

1. The cost of a railroad ticket depends upon the number of miles to be traveled. That is, the cost is a function of the distance. The distance is the independent variable, and the cost is the dependent variable. Likewise, the distance one can ride depends upon the cost of the ticket, that is, the distance is a function of the cost. In the latter way of looking at it, the cost is the independent variable and the distance the dependent variable.

2. If a train moves uniformly, the distance traversed is a function of the time.

3. An iron bar expands when heated. Its length is a function of the temperature.

4. $3x-2$ is a function of x because its value depends upon that of x.

285. "Function of x" is often briefly expressed by $f(x)$.

Thus, if $f(x)=5x-4$,

then, $f(2)=5\cdot 2-4=6$,

$f(0)=5\cdot 0-4=-4$,

and $f(r)=5r-4$.

286. Graphs of Functions. The **graph of a function** is a diagram representing the variation of the function due to the variation in the value of its variable.

Thus, the first graph in Sec. 271, p. 200, shows that the value of $f(x)=4x$ varies from 0 to 8 as x varies from 0 to 2.

287. Since the graph of the function $ax+b$ is a straight line (Sec. 273), this function is called a **linear function,** and the corresponding equation, $y=ax+b$, a linear equation.

288. Graphs of Incomplete Equations. Either x or y may be lacking in an equation to be plotted.

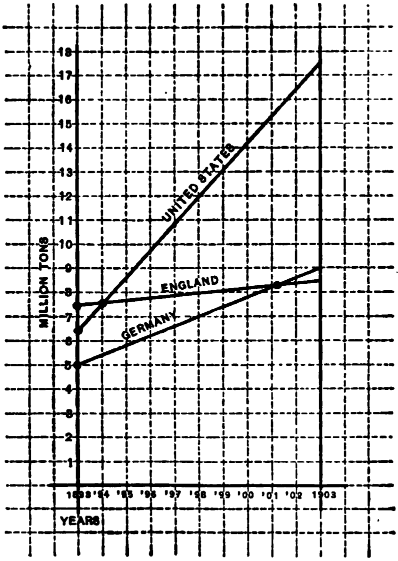
Thus, the equation $y = 3$ means $y = 0x + 3$, in which $y = 3$ for all values of x; consequently the graph of $y = 3$ is a line parallel to the x-axis and 3 units above it. Similarly, the graph of $x = 2$ is a line parallel to the y-axis, and 2 units to the right of it.

WRITTEN EXERCISES

Construct the graphs of:

1. $x + y = 1$.
2. $x + y = 0$.
3. $2y - 4x = 2$.
4. $x - y = 2$.
5. $x = 4y - 1$.
6. $3x + 2y = 6$.
7. $0 = x + 3$.
8. $2x - 2y = 4$.
9. $3x + 3y = 9$.
10. What line must the graph cross when $x = 0$? When $y = 0$? Construct the graph of $y = 2x - 3$ by using the points for which $x = 0$, and $y = 0$, respectively.

GRAPHS OF EQUATIONS WITH TWO UNKNOWNS



289. Preparatory.

1. The diagram shows the general trend in the increase of the pig-iron product in the United States, Germany, and England during a period of ten years.

(*a*) In which country has the increase been the most rapid? The least rapid?

(*b*) In what year was the amount produced in England and the United States the same?

(*c*) Answer the same question for England and Germany.

2. A factory has a fixed charge of $20 daily, and makes an average gross profit of $2 daily per workman employed. What remains after the fixed charge is paid is net profit. Make a graphic representation of how the net profit (p) varies while the number of workmen (w) varies from 0 to 50.

The graph exhibits the change in p due to a change in w subject to the above relation; or it is the graph of the equation $p = 2w - 20$.

3. A second factory has a fixed charge of $60, and makes an average gross profit of $3 daily per workman employed. Construct a graph to represent the net profits of the second factory in the diagram made for Exercise 2.

What is the relation between p and w, of which this is the graph?

4. From the diagram read:

(*a*) The number of workmen for which each factory makes the same net profit.

(*b*) The amount of the profit mentioned in (*a*).

(*c*) The number of workmen beyond which the second factory makes the larger net profit.

(*d*) The net profit of each factory where 25 workmen are employed. 40 workmen. 15 workmen. 10 workmen. 20 workmen.

(*e*) The number of workmen each factory must employ to make $30 net profit. Also $60 net profit. No net profit.

290. Graphs may be employed to represent the solution of any set of two simultaneous linear equations with two unknowns.

For example:

$$\begin{cases} x - y = 1, & (1) \\ x + 2y = 4. & (2) \end{cases}$$

We have already seen how to represent graphically all of the solutions of a single equation of the first degree in two unknowns.

If we represent the equation $x - y = 1$, and the equation $x + 2y = 4$, in the same diagram it is possible at once to read the values of x and y that satisfy both equations.

Table of Values for Equation (1)

x	y
0	-1
1	0
2	1

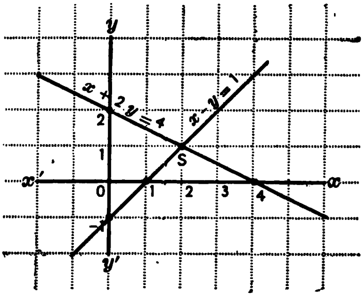


Table of Values for Equation (2)

x	y
0	2
1	$1\frac{1}{2}$
2	1

ORAL EXERCISES

1. Read from the graph of the first equation the value of y when $x = 0$. When $x = 1$. When $x = 2$. When $x = 3$.

2. How many points are needed to fix the position of a straight line?

3. Read from the graph of the second equation the value of y when $x = 0$. When $x = 2$. When $x = 4$.

4. What point in the diagram is common to the two graphs? What are the values of x and y for point s?

5. What point in the diagram represents the solution of the system of equations?

291. *The solution of two simultaneous equations of the first degree in two unknowns is represented graphically by the point of intersection of the graphs of the equations.*

Note. Since the accuracy of the graphical solution depends upon the precision of the diagram, the results so found must be regarded as approximate. Their accuracy must be tested in the usual way. The graphical method of solution is of practical use chiefly in applied mathematics where an approximate result is commonly sufficient, but in theoretic mathematics it is important as exhibiting clearly to the eye the relations between the variables involved.

WRITTEN EXERCISES

Solve graphically:

1. $x + y = 5$,
   $3x - 2y = 0$.
2. $3x - y = 1$,
   $x + 2y = 12$.
3. $x + y = 0$,
   $y + 3 = 4$.
4. $x + 2y = 6$,
   $2x - y = 2$.
5. $x + y = 7$,
   $2x - y = 5$.
6. $3x + y = 12$,
   $x - y = 0$.
7. $x - y = 1$,
   $2x - 3y = 1$.
8. $x + 3y = 8$,
   $x - 2y = -2$.
9. $2x - y = 1$,
   $x - 2y = -4$.
10. $4x + y = 6$,
    $2x - y = 0$.
11. $x + 3y = 12$,
    $3x - y = 6$.
12. $x - \frac{1}{2}y = 2$,
    $\frac{1}{3}x + y = 3$.

SUMMARY

1. In graphical work what are the *axes?* The *quadrants?* The *coördinates?* The *origin?* The *abscissas?* The *ordinates?* Secs. 276–280.

2. What is a *variable?* A *constant?* Secs. 281, 282.

3. What is a *function?* What is meant by *dependent* and *independent variables?* Sec. 284.

4. What does the graph of a function show? Sec. 286.

5. When two linear equations are plotted on the same axes, what does their intersection represent? Sec. 291.

HISTORICAL NOTE

We have stated in an earlier note that the chief symbols of algebra were perfected during the sixteenth century, and that scholars had then discovered many properties of algebraic expressions as well as methods of solving equations. But at the beginning of the seventeenth century a new channel for algebraic study was opened by the French philosopher and mathematican, René Descartes, who laid the foundation for what we now call "graphical algebra."

René Descartes, was born at La Haye in 1596. At the age of twenty-one he enlisted as a soldier under Prince Maurice of Orange, pursuing the study of mathematics in his leisure hours. In 1829 he went to Holland, Dutch culture and learning then being at its height, and there devoted twenty years to the preparation of his works in mathematics and philosophy His belief in the certainty and accuracy of the reasoning used

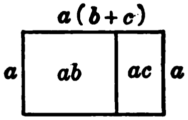
René Descartes

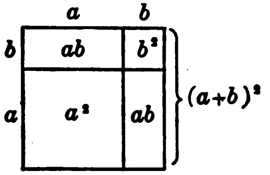
in arithmetic and algebra led him to apply the same method to physics and other sciences. Descartes adopted the plan of locating points in a plane by means of their distances from two fixed lines or axes, and was the first to plot lines representing algebraic expressions, as explained in Chapter XVIII. The words "ordinate" and "coördinates" were introduced by him, probably from the Latin phrase *lineæ ordinatæ*, used by the Roman surveyors to express parallel lines. The word *abscissa* was first used by an Italian writer (1659).

Graphical algebra, in a broader sense, was used many centuries before Descartes. For example, the products:

$$a(b+c) = ab + ac \text{ and } (a+b)^2 = a^2 + 2ab + b^2,$$

which were not considered direct results of multiplication before Diophantos, were represented graphically by Euclid (300 B.C.) in the following way:





The first step toward constructing graphs of equations was taken by the Hindoos in Bhaskara's time, when they represented positive and negative numbers by opposite segments on a straight line; from this conception it was only one step more to the expression of the relation of one variable to another by measurements along two lines, or by coördinates. This was done by Nicholas Horem, a French teacher in Paris in the fourteenth century. Naturally he began with positive numbers and constructed graphs in the first quadrant only, so it remained for Descartes to show how the changes in a function of x can be shown graphically in the four quadrants of a plane for all real values of x.

CHAPTER XIX

SYSTEMS OF LINEAR EQUATIONS

EQUATIONS WITH TWO UNKNOWNS

292. Problems involving more than one unknown often require more than one equation in their solution.

EXAMPLES

1. A street car of a certain make has 48 seats, some single and some double; the seating capacity of the car is 56. How many single seats are there? How many double seats?

SOLUTION. Let $x =$ the number of single seats,
and $y =$ the number of double ones.

Then, $$\begin{cases} x + y = 48, & (1) \\ x + 2y = 56. & (2) \end{cases}$$

Subtracting equation (*1*) from equation (*2*), $y = 8.$ (*3*)

Substituting this value of y in equation (*1*), $x + 8 = 48.$ (*4*)

$\therefore x = 40.$ (*5*)

Therefore there were 40 single seats and 8 double ones. (*6*)

TEST. $$\begin{cases} 40 + 8 = 48. \\ 40 + 2 \cdot 8 = 56. \end{cases}$$

2. Solve: $$\begin{cases} 10x + 5y = 45, & (1) \\ 6x + 8y = 47. & (2) \end{cases}$$

Multiplying (*1*) by 3, $30x + 15y = 135.$ (*3*)

Multiplying (*2*) by 5, $30x + 40y = 235.$ (*4*)

Subtracting (*3*) from (*4*), $25y = 100$, and $y = 4.$ (*5*)

Substituting $y = 4$ in (*1*), $x = 2\frac{1}{2}.$ (*6*)

Test the values $x = 2\frac{1}{2}$ and $y = 4$ as in equation (*2*).

Where in the solution was one unknown made to disappear, and how?

WRITTEN EXERCISES

Solve for x and y:

1. $x + y = 4,$
   $3x + 5y = 18.$
2. $x + y = 7,$
   $3x + 10y = 42.$
3. $x + y = 5,$
   $2x + 5y = 22.$
4. $x + y = 0,$
   $5y + 2x = -3.$
5. $2x + y = 4,$
   $x + 2y = 5.$
6. $3x + 3y = 0,$
   $x + 4y = 3.$
7. $x - y = 1,$
   $2x + y = 5.$
8. $x - y = 1,$
   $3x + 2y = 13.$
9. $x - 2y = -4,$
   $2x - y = 1.$
10. $3x - 3y = 0,$
    $x + y = 6.$
11. $4x - 2y = 0,$
    $x + 3y = 7.$
12. $5x + 3y = 23,$
    $3x + 5y = 17.$
13. $5x - 3y = 11,$
    $3x - y = 5.$
14. $4x + y = -7,$
    $y - 4x = 9.$
15. $2x - y = 0,$
    $y + 2x = 4.$
16. $10y + x = 11,$
    $y + 10x = 11.$
17. $5y - x = 3,$
    $y - 5x = -9.$
18. $4y - 4x = 8,$
    $y + x = 8.$
19. $2x + y = 9,$
    $x + 2y = 12.$
20. $7x + y = 14,$
    $x + 9y = 64.$
21. $2x + 3y = 22,$
    $5x + 4y = 48.$
22. $2x + 6y = 34,$
    $6x + 8y = 62.$
23. $2x + 4y = 38,$
    $3x + y = 27.$
24. $x + 10y = 11,$
    $\frac{1}{2}x + \frac{1}{5}y = \frac{7}{10}.$
25. $\frac{1}{2}x - \frac{1}{2}y = 6,$
    $x - 2y = 0.$
26. $5x - 3y = 1,$
    $2x + y = 7.$
27. $2x + 3y = 2,$
    $8x - 6y = 2.$
28. $2x - 3y = 4,$
    $7x - 6y = 3.$

293. In the work of solving the preceding problems the process has always been to combine the given equations so as to obtain an equation involving only one of the unknowns.

The other unknown is said to have been **eliminated.**

294. Independent Equations. Equations which express different relations between the same unknowns are called **independent equations.**

Thus, $x - y = 1$ and $x + y = 7$ are independent equations; for one expresses the *difference* between two numbers, x and y, while the other expresses the *sum* of the same two numbers.

$x - y = 1$ and $2x - 2y = 2$ are not independent, for the second when divided by 2 is the same as the first.

295. While a linear equation in one unknown has but one root, a linear equation in two unknowns is satisfied by any number of sets of values of the unknowns. In the former the unknown is a *constant*, while in the latter the unknowns are *variables.*

296. Simultaneous Equations. Two or more equations are said to be **simultaneous** when all of them are satisfied by the same values of the unknowns.

297. Systems of Equations. Two or more equations considered together are called a **system** of equations.

Thus, each of the exercises of p. 212 contains a system of equations.

298. A system of independent simultanous linear equations in which the number of equations is the same as the number of unknowns is satisfied by only one set of values of the unknowns.

299. Method of Addition and Subtraction. All systems of two independent simultaneous equations of the first degree in two unknown quantities can be solved by the **method of addition and subtraction.**

The method of addition and subtraction consists in multiplying one or both of the given equations by such numbers that the coefficients of one of the unknowns become equal. Then by subtraction this unknown is eliminated, and the solution is reduced to that of a single equation.

If the coefficients of one unknown are made numerically equal, but have opposite signs, the equations should be added.

EXAMPLE

Solve: $\begin{cases} 4x - 3y = 5, & (1) \\ 6x + 2y = 14. & (2) \end{cases}$

Multiplying (*1*) by 2, $8x - 6y = 10,$ (*3*)

and (*2*) by 3, $18x + 6y = 42.$ (*4*)

Adding (*3*) and (*4*), $26x = 52,$ (*5*)

and $x = 2.$ (*6*)

Substituting 2 for x in (*2*), $y = 1.$ (*7*)

Test. Substitute $x = 2$, $y = 1$ in equations (*1*), (*2*).

WRITTEN EXERCISES

Solve and test:

1. $2x + y = 5,$
   $x - y = 1.$
2. $3x + 2y = 7,$
   $x - 2y = 3.$
3. $2x + y = 1\frac{1}{2},$
   $x - y = 0.$
4. $3x + 2y = 2,$
   $x + y = \frac{5}{6}.$
5. $x + 5y = 35,$
   $5x + y = 31.$
6. $4x + 3y = 18,$
   $2x + 2y = 10.$
7. $x + 2y = 0,$
   $4x - 3y = 4.$
8. $2x + 3y = 4,$
   $3x + 2y = 1.$
9. $\frac{1}{2}x + \frac{1}{3}y = 4,$
   $\frac{3}{4}x + \frac{2}{3}y = 5.$
10. $.3x + .2y = .1,$
    $.2x + .3y = .4.$
11. $4x + y = 34,$
    $4y + x = 16.$
12. $4x - y = 7,$
    $3x + 4y = 29.$
13. $2x + 3y = 4,$
    $3x - 2y = -7.$
14. $2x + 3y - 8 = 0,$
    $7x - y - 5 = 0.$

15. Find two numbers whose sum is 13 and the difference between twice the first and three times the second is 3.

16. The cost of a house and lot was $6500; the house cost $3500 more than the lot. What was the cost of each?

17. In a recent year the value of the hay crop in the United States exceeded that of the cotton crop by $30 millions; the two amounted to $1180 millions. Find the value of each.

18. The value of the cotton crop in a certain year exceeded that of the wheat crop by $50 millions; the two amounted to $1100 millions. What was the value of each?

19. If the value of the sugar produced, in a given year, had been increased by $\frac{4}{25}$ of itself, it would have equaled the value of the barley crop; the two amounted to $108,000,000. What was the value of each?

300. Method of Substitution. The method of addition and subtraction can be used to solve all systems of two simultaneous equations with two unknowns, but occasionally problems occur in which other methods are shorter. The most useful of these is the **method of substitution.**

EXAMPLES

1. Solve: $\begin{cases} 3x+2y=6, & (1) \\ 2x-y=4. & (2) \end{cases}$

From equation (2), $y=2x-4.$ (3)

Substituting in (1), $3x+4x-8=6.$ (4)

$\therefore\ 7x=14,$ (5)

and $x=2.$ (6)

From (6) and (3), $y=0.$ Verify. (7)

2. Solve: $\begin{cases} 4x-3y=9, & (1) \\ 5x=15. & (2) \end{cases}$

From equation (2), $x=3.$ (3)

Substituting in (1), $12-3y=9.$ (4)

$\therefore\ y=1.$ Verify. (5)

301. *The method of substitution consists in expressing one unknown in terms of the other by means of one equation and substituting this value in the other equation, thus eliminating one of the unknowns.*

This may be the shorter method when an unknown in either equation has the coefficient 0, +1, or −1.

WRITTEN EXERCISES

Solve by substitution:

1. $x+y=75,$ $3x-3y=15.$
2. $5x+2y=31,$ $x=12y.$
3. $3x+2y=12,$ $x+y=5.$
4. $x+2y=7,$ $x=\frac{3}{2}y.$
5. $x+3y=11,$ $3x+y=9.$
6. $x-4y=8,$ $x+2y=14.$

7. $y + 15x = 53,$
$x + 3y = 27.$

8. $5x - 7y = -35\frac{1}{2},$
$2x - y = \frac{5}{7}.$

9. $3x - 5y = 31,$
$4y = -10x.$

10. $x + y = \frac{7}{6},$
$x + 7y = \frac{31}{6}.$

11. $x + y = 12,$
$3x + y = 24.$

12. $x + 2y = 10,$
$5x - 2y = 2.$

Solve and test:

13. $x + y = 29,$
$2x + 5y = 103.$

14. $x + y = 480,$
$12x + 20y = 7520.$

15. $x = 4y + 76,$
$x - y = 430.$

Show that each of the following exercises can be solved by using one equation with one unknown, or by using two equations with two unknowns:

16. The continued height of a tower and flagstaff is 100 ft.; the height of the tower is 60 ft. more than the length of the staff; find the height of each.

17. A house and lot are worth $3500; the house is worth $2500 more than the lot; find the value of each.

18. The area of the United States and the British Isles together is 3,146,600 sq. mi.; the area of the United States diminished by 600 sq. mi. is 25 times that of the British Isles. What is the area of each?

19. Japan's exports in a recent year plus its exports 10 yr. ago were approximately $180,000,000; this was a gain of 400% on their value 10 yr. before. What was their value at that time?

20. It is estimated that the part of the population of the United States living on farms is $\frac{2}{3}$ of the rest of the population. Taking the total to be 93 millions, how many live on farms?

21. The average creamery of the Eastern States produces only $\frac{3}{5}$ as much butter as the average creamery of the Western States; two average eastern creameries and three average western creameries together produce 42,000 lb. annually. What is the annual output of each?

302. General Form of Linear Equations with Two Unknowns. A *general form* for an equation of the first degree with two unknowns is

$$ax + by = e.$$

A *general form* for two such equations is

$$\begin{cases} ax + by = e & (1) \\ cx + dy = f. & (2) \end{cases}$$

303. General Solution. From the general equations (*1*) and (*2*) it is possible (without knowing the values of a, b, c, d, e, f) to find a *general form* for the solution.

Solve:

$$\begin{cases} ax + by = e, & (1) \\ cx + dy = f. & (2) \end{cases}$$

Multiplying (*1*) by c and (*2*) by a,
$$cax + cby = ce, \quad (3)$$
$$acx + ady = af. \quad (4)$$

Subtracting (*3*) from (*4*),
$$ady - cby = af - ce. \quad (5)$$

Thus,
$$(ad - bc)y = af - ce. \quad (6)$$

Dividing by the coefficient of y,
$$y = \frac{af - ce}{ad - bc}. \quad (7)$$

Eliminating by multiplying (*1*) by d, and (*2*) by b, and subtracting,
$$x = \frac{de - bf}{ad - bc}. \quad (8)$$

From these values of x and y the values in any particular case can be found by substituting for a, b, c, d, e, f, their particular values. But there are other forms of these values better adapted to substitution, and these are given in Chapter XXV.

304. Literal Equations. The general solution of Section 303 shows that the processes of elimination previously explained serve fully to solve systems of simultaneous equations with *literal coefficients.*

WRITTEN EXERCISES

Solve:

1. $12x + 6y = -18$,
   $48x - 9y = 60$.

2. $x + 17y = 300$,
   $11x - y = 104$.

3. $2x - 3y = 5a - b$,
   $3x - 2y = b + 5a$.

4. $26x + 42y = 33$,
   $39x + 28y = 44$.

5. $3x - 5y = -\frac{1}{8}$,
$6x + 2y = 3\frac{7}{8}$.

6. $12x + 6y = 13$,
$48x - 9y = 30$.

7. $5x + 45y = 145$,
$15x - 9y = 3$.

8. $\frac{x}{a} + \frac{y}{b} = 1$,
$\frac{x}{2a} - \frac{y}{3b} = 4$.

9. Solve for a and l:
$l = a + (n-1)d$.
$s = \frac{n}{2}(a + l)$.

10. $3p + q = 3$,
$5p - q = 13$.

11. $4h - 2b = 1$,
$3h + b = 5$.

12. $4v - 5w = 6$,
$-3v + 4w = 1$.

13. $14R + 3r = 2$,
$9R + 2r = 6$.

14. $(a+c)x - (a-c)y = 2ab$,
$(a+b)x - (a-b)y = 2ac$.

15. $(a+1)x - (b+1)y = c$,
$(a-1)x + (b-1)y = d$.

16. $(m-n)x + (m+n)y = mn$,
$(m-p)x + (m+p)y = mp$.

17. An investor purchases two kinds of securities; one kind pays 2 % and the other 4 % annually; his annual income from both sources is \$900; if he had invested as much in the 2 % securities as he did in the 4 per cents, and *vice versa*, his income would have been \$600. Find his investment in each.

SOLUTION. 1. Let x and y be the number of dollars invested at 2 % and 4 %, respectively.

2. Then, by the conditions, $.02x + .04y = 900$,
3. and $.04x + .02y = 600$.
4. Multiplying (2) by 2, $.04x + .08y = 1800$.
5. Subtracting (3) from (4), $.06y = 1200$.
6. Solving (5), $y = 20{,}000$.
7. Substituting $y = 20{,}000$ in (1), $.02x + 800 = 900$.
8. Solving (7), $x = 5000$.

∴ he invested \$5000 at 2 % and \$20,000 at 4 %.

TEST. 2 % of \$5000 + 4 % of \$20,000 = \$900;
2 % of \$20,000 + 4 % of \$500 = \$600.

18. An investor purchased Pennsylvania Railroad stock paying 6 % annually, and municipal bonds paying 4 %; his annual income from both was \$2100; if the stock had paid 1 % less and the bonds 1 % more, his total income would have been \$2000. How much did he invest in each?

19. An investment of \$20,000 in one stock, and one of \$10,000 in another, together yield \$1300 annually; an investment of half as much in the first and twice as much in the second would together yield \$1100 annually. What is the annual rate of dividend in each stock?

20. A standard daily ration for an adult laborer requires 4 oz. of protein and 4 oz. of fat.

The following table shows the approximate amounts of protein and fat in various foods:

FOOD	PER CENT OF FAT	PER CENT OF PROTEIN
Mutton	37	14
Pork (fresh)	26	13
Eggs	9	13
Bread (white)	1	9
Beans (dried)	2	22
Corn (green)	1	3
Rice	½	8

How many ounces of mutton and bread are needed to make a standard ration for one day?

SOLUTION. 1. Let x and y be the numbers of ounces required of mutton and bread, respectively.

2. Then $0.14x + 0.09y$ is the amount of protein in these foods according to the table.

3. Also $0.37x + 0.01y$ is the amount of fat in these foods.

4. Thus, $0.14x + 0.09y = 4,$

5. and $0.37x + 0.01y = 4.$

6. Multiplying (4) by 100, $14x + 9y = 400.$

7. Multiplying (5) by 100, $37x + y = 400.$

8. Subtracting (7) from (6), $23x - 8y = 0. \quad \therefore x = \frac{8}{23}y.$

9. Using (8) in (7), $\frac{37 \cdot 8y}{23} + y = 400.$

10. $\therefore y = 28.8$ (to one decimal place) and $x = 10.0+$

11. $\therefore$ the ration is 10 oz. of mutton and 28.8 oz. of bread.

A negative result for either unknown quantity would show that it is impossible to make up the standard ration out of the foods named.

Find which of the following combinations of foods can make a standard ration, and the number of ounces of each food required:

21. Mutton and beans.
22. Mutton and rice.
23. Bread and eggs.
24. Pork and beans.
25. Eggs and corn.
26. Mutton and corn.
27. Bread and pork.
28. Bread and rice.
29. Pork and rice.
30. Eggs and rice.

31. A man has a certain sum of money invested at 5 %; he reinvests the whole sum, placing three times as much of it at 8 % as he does at 4 %. His income is increased $ 200 a year by the change. How much money has he, and what is his income?

305. Solving Fractional Equations. I. *In case of fractional equations, when the unknown quantities occur only in monomial denominators, it is best not to clear of fractions.*

EXAMPLES

1. Solve:
$$\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}=2. & (1) \\[2ex] \dfrac{1}{2x}-\dfrac{1}{3y}=\dfrac{5}{36}. & (2) \end{cases}$$

Multiplying (2) by 4, $$\frac{4}{2x}-\frac{4}{3y}=\frac{20}{36}. \quad (3)$$

Simplifying (3), $$\frac{2}{x}-\frac{4}{3y}=\frac{5}{9}. \quad (4)$$

Subtracting (4) from (1), $$\frac{13}{3y}=\frac{13}{9}. \quad (5)$$

Dividing (5) by 13, $$\frac{1}{3y}=\frac{1}{9}. \quad (6)$$

Clearing (6) of fractions, $$3y=9. \quad (7)$$

$$\therefore y=3. \quad (8)$$

Substituting 3 for y in (1), $$x=2. \quad (9)$$

TEST. $$\frac{2}{2}+\frac{3}{3}=2 \text{ and } \frac{1}{2\cdot 2}-\frac{1}{3\cdot 3}=\frac{5}{36}.$$

It is evident that the above equations, if cleared of fractions, would contain terms in xy which would complicate the solution.

2. In such equations the unknown quantities may be thought of as $\frac{1}{x}$ and $\frac{1}{y}$.

Thus, to solve $\frac{10}{x}+\frac{9}{y}=5$, and $\frac{35}{x}-\frac{6}{y}=5$, is to solve

$$10\left(\frac{1}{x}\right)+9\left(\frac{1}{y}\right)=5, \text{ and } 35\left(\frac{1}{x}\right)-6\left(\frac{1}{y}\right)=5.$$

The coefficients are now manipulated as in the case of integral equations. It is often convenient to introduce new unknowns in such problems; in this case, for example, by putting

$\frac{1}{x}=x'$ and $\frac{1}{y}=y'$, the equations become $$10x'+9y'=5,$$ $$35x'-6y'=5.$$

306. The same principles apply in exactly the same way to equations with *literal coefficients.*

EXAMPLE

1. Solve:
$$\begin{cases}\frac{a}{x}+\frac{b}{y}=m, & (1)\\ \frac{c}{x}+\frac{d}{y}=n. & (2)\end{cases}$$

Regard $\frac{1}{x}$, $\frac{1}{y}$ as the unknowns, multiply (1) by d and (2) by b and subtract.

Then, $$\frac{ad}{x}-\frac{bc}{x}=md-nb,$$

or, $$(ad-bc)\frac{1}{x}=md-nb.$$

Hence, $$\frac{1}{x}=\frac{md-nb}{ad-bc}. \qquad (3)$$

Similarly, multiply (1) by c and (2) by a and subtract.

Then, $$(ad-bc)\frac{1}{y}=an-mc,$$

and $$\frac{1}{y}=\frac{an-mc}{ad-bc}. \qquad (4)$$

From (3), $$x=\frac{ad-bc}{md-bn}. \qquad (5)$$

From (4), $$y=\frac{ad-bc}{an-mc}. \qquad (6)$$

Test. In literal equations it is generally more convenient to test by reworking in a different way, when possible, than by substitution.

WRITTEN EXERCISES

Solve for x and y:

1. $\frac{x}{5}+y=3,$

   $x+\frac{y}{2}=11.$

2. $\frac{x}{3}+\frac{3y}{2}=4,$

   $\frac{5x}{6}-2y=-13.$

3. $\frac{6}{x}+5y=18,$

   $\frac{8}{x}-y=1.$

4. $\frac{x}{3}+\frac{3y}{4}=-4,$

   $\frac{x}{y}=\frac{3}{4}.$

5. $x+y=5,$

   $\frac{x}{y}=\frac{1}{4}.$

6. $x+\frac{3}{4}-\frac{x}{3}-\frac{5y}{6}=\frac{5x}{9}+\frac{17}{36},$

   $\frac{1}{7}(y-3x)+\frac{1}{8}(9x-3y)=12.$

7. $\frac{20}{x}+\frac{4}{3y}=0,$

   $\frac{x}{2}-\frac{y}{5}=4.$

8. $\frac{x+1}{y}=1,$

   $\frac{y+1}{x}-1=\frac{3}{2}.$

9. $\frac{7x}{5}-\frac{5y}{6}=2,$

   $\frac{3x}{10}-\frac{5y}{12}=4.$

10. $\frac{1}{x}+\frac{1}{y}=\frac{3}{4},$

    $\frac{1}{x}-\frac{1}{y}=\frac{1}{4}.$

11. $\frac{6}{x}+\frac{8}{y}=3,$

    $\frac{12}{x}-\frac{20}{y}=-3.$

12. $\frac{1}{3x}+\frac{1}{2y}=-\frac{1}{6},$

    $\frac{1}{x}+\frac{2}{y}=-1.$

13. $\frac{3}{x}-\frac{2a}{y}=b,$

    $\frac{5}{2x}+\frac{3b}{y}=c.$

14. $2x+\frac{1}{y}=14,$

    $4x-\frac{5}{y}=14.$

SUGGESTION. Regard x and $\frac{1}{y}$ as the unknowns.

15. $\frac{6}{x}-2y=6,$

    $\frac{9}{x}+y=1.$

16. $\frac{5}{x}+3y=17,$

    $\frac{2}{x}-2y=-6.$

17. $\frac{1}{x}+\frac{a}{y}=2,$
$\frac{b}{x}-\frac{1}{y}=a.$

18. $\frac{2a}{x}-\frac{3b}{y}=4,$
$\frac{3a}{x}-\frac{4b}{y}=2c.$

19. $\frac{3}{x}+\frac{2}{y}=-12,$
$\frac{4}{x}-\frac{3}{y}=1.$

20. $\frac{1}{x}+\frac{a}{y}=1,$
$\frac{a}{x}+\frac{1}{y}=1.$

21. The sum of the reciprocals of two numbers is $\frac{5}{4}$, and the difference of the reciprocals is $\frac{1}{4}$. Find the numbers.

22. Find two numbers such that 3 times the reciprocal of the first added to 5 times the reciprocal of the second makes 2, and 24 times the reciprocal of the first diminished by 10 times the reciprocal of the second makes 1.

307. II. *It is usually best to clear of fractions when the unknowns occur in polynomial denominators.*

EXAMPLES

1. Solve for x and y:

$$\begin{cases} \frac{x+1}{y+1}=\frac{1}{2}, & (1) \\ \frac{x-1}{y-1}=\frac{1}{4}. & (2) \end{cases}$$

Clearing (*1*) of fractions, $2x+2=y+1$, or $2x-y=-1$. (*3*)

Clearing (*2*) of fractions, $4x-4=y-1$, or $4x-y=3$. (*4*)

Solving (*3*) and (*4*), $x=2$ and $y=5$. (*5*)

2. Solve:

$$\begin{cases} \frac{2x+1}{x-4}=\frac{2y+18}{y+6}, & (1) \\ \frac{x+1}{y}=-2. & (2) \end{cases}$$

Clearing (*1*) of fractions, $2xy+y+12x+6=2xy+18x-8y-72.$ (*3*)

Collecting terms in (*3*), $-6x+9y=-78.$ (*4*)

From (*2*), $x+2y=-1.$ (*5*)

Solving (*4*) and (*5*), $x=7,\ y=-4.$ (*6*)

3. Solve:

$$\begin{cases} \dfrac{2x+6}{x}=y+\dfrac{10x+6}{x}, & (1) \\[2ex] \dfrac{x+4}{y+3}=-2. & (2) \end{cases}$$

Separating the fractions of (1), $\dfrac{2x}{x}+\dfrac{6}{x}=y+\dfrac{10x}{x}+\dfrac{6}{x}$. (3)

Simplifying the terms of (3), $2+\dfrac{6}{x}=y+10+\dfrac{6}{x}$. (4)

Hence, $2=y+10$. (5)

Or, $y=-8$. (6)

Substituting $y=-8$ in (2), $x=6$. (7)

WRITTEN EXERCISES

Solve:

1. $\dfrac{2x+3y}{1+4y}=1,\quad \dfrac{4x-2y}{x-y}=-2.$

2. $\dfrac{x+1}{y}=\dfrac{x+5}{y+2},\quad \dfrac{2x-3y}{4x-2y}=\dfrac{1}{2}.$

3. $\dfrac{x+1}{y}=4,\quad \dfrac{y+1}{x}=3.$

4. $\dfrac{x}{y}=\dfrac{5}{4},\quad \dfrac{x+1}{y+1}=\dfrac{16}{13}.$

5. $\dfrac{x+5}{y-3}=\dfrac{x+1}{y-2},\quad \dfrac{x+2}{y-3}=3.$

6. $\dfrac{x+1}{5y}=-1,\quad \dfrac{3x+2y}{y+6}=2.$

7. $y+1+\dfrac{1}{3x}=2y-\dfrac{27x-3}{9x},\quad \dfrac{1}{3}+\dfrac{y}{2x}=\dfrac{x}{3}-\dfrac{30x-5y}{10x}.$

8. $\dfrac{c-2y}{a+b-2x}\cdot\dfrac{c}{b-a}=-1,\quad \dfrac{c-2y}{b-2x}\cdot\dfrac{c}{b}=-1.$

9. $\dfrac{y}{x+a}\cdot\dfrac{c}{a}=1,\quad \dfrac{y}{x-a}=-\dfrac{c}{a}.$

10. $\dfrac{2}{x}+\dfrac{1}{y}=2,\quad \dfrac{3}{x}-\dfrac{2}{y}=5.$

11. $\dfrac{2x}{3}-\dfrac{5y}{12}-\left(\dfrac{3x}{2}-\dfrac{4y}{3}\right)=-\dfrac{2}{3},\quad \dfrac{x-y}{x+y}=\dfrac{1}{5}.$

12. $3x-5ay=8ab,\quad \dfrac{x}{7a}-7y+3b=0.$

13. $\frac{2x}{3} - 4 + \frac{y}{2} + x = 8 - \frac{3y}{4} + \frac{1}{12}, \quad \frac{y}{6} - \frac{x}{2} + 2 = \frac{1}{6} - 2x + 9\tfrac{5}{6}.$

14. $\frac{x+y}{x-y} = 4, \quad 3x + 5y = \frac{3}{4}.$

15. $\frac{3x+12y}{1+x} = 3, \quad \frac{x+4y}{1+4y} = -\frac{1}{4}.$

16. $\frac{4(3x-2y)}{5(x+2y)} = \frac{8}{10}, \quad \frac{1}{3x} + \frac{3}{5} = 0.$

TYPES OF LINEAR SYSTEMS

308. Certain systems of equations have more than one solution.

EXAMPLES

1. Solve: $3x + y = 5.$

We may express x in terms of y, or express y in terms of x, but neither is eliminated. To obtain sets of numerical values for both of them we may give arbitrary values to one, and find the corresponding values for the other. In the above equation, if we give x the values,

$-2, -1, 0, 1, 2$; and so on.
y will have the values, $11, 8, 5, 2, -1$; and so on.

2. Solve: $\begin{cases} x + y = 3, \\ 2x + z = 5. \end{cases}$

This is a system of two equations with three unknowns. We can dispose of only one of them by the methods of elimination, hence the result is an equation in two unknowns which may have any number of solutions as in the case of Exercise 1. Thus, the given system has an unlimited number of solutions.

309. Indeterminate Systems. Systems of equations like those in Section 308, admitting an unlimited number of solutions, are called *indeterminate systems.*

310. The equations of a system may be dependent.

EXAMPLE

Solve: $\begin{cases} x - 2y = 1, \\ 3x - 6y = 3. \end{cases}$

If the first equation is multiplied by 3, the result is the second equation. Hence, nothing is gained by attempting to eliminate one of the unknowns.

311. Dependent Equations. If, in a system of equations, two or more equations express the same relation between the unknowns, the equations are said to be *dependent.*

In such a case, if the number of unknowns is the same as the number of equations, the system is indeterminate.

312. The equations of a system may be inconsistent.

EXAMPLE

Solve:
$$\begin{cases} x+2y=3, \\ 2y+x=4. \end{cases}$$

The first equation asserts that $x+2y=3$, but the second equation asserts that $x+2y=4$. Evidently both of these relations cannot be true for the same values of x and y. If equation (1) is subtracted from equation (2), the result, $0=1$, shows that they are incompatible.

313. Inconsistent Equations. Equations which express contradictory relations between the unknowns are called *inconsistent,* or *incompatible, equations.*

314. Number of Solutions. Two linear equations in two unknowns may be:

1. Determinate and have one solution.
2. Dependent, and have an unlimited number of solutions.
3. Contradictory and have no solution.

ORAL EXERCISES

Classify the following systems according to Secs. 308–313:

1. $\begin{cases} x+y=2, \\ x+y-z=3. \end{cases}$

2. $\begin{cases} 2x-y=4, \\ 4x-2y=12. \end{cases}$

3. $\begin{cases} 3x-y=2, \\ 9x-3y=6. \end{cases}$

4. $\begin{cases} 2x-y=7, \\ x+y=5. \end{cases}$

5. $\begin{cases} ax-by=c, \\ ax-by=bc. \end{cases}$

6. $\begin{cases} 5x-y=4, \\ 10x-8=2y. \end{cases}$

INTERPRETATION

315. Problems may lead to equations whose solutions are inconsistent with the given conditions. Hence, any solutions that do not admit of interpretation should be rejected.

EXAMPLES

1. A shelf contained 20 books; some were histories and the rest biographies. If 4 times the number of biographies, less 2 times the number of histories, was 35, how many were there of each kind?

Solution. $x + y = 20,$ (1)

$4y - 2x = 35.$ (2)

Multiplying (1) by 2 and adding, $6y = 75.$

Therefore, $y = 12\frac{1}{2}$, and $x = 7\frac{1}{2}$.

Discussion. The equations are correctly written and solved, but the values show that the conditions of the problem are impossible, because we cannot have a fractional number of books.

2. If the sum of the length and width of a rectangular garden is 5 ft. and the difference between the dimensions is 19 ft., find the length and width of the garden.

Solution. $x + y = 5,$ (1)

$x - y = 19.$ (2)

Adding (1) and (2), $2x = 24$, and $x = 12.$ (3)

Substituting x in (2), $y = -7.$ (4)

Discussion. The equations are correctly written and solved; hence, the statement of the problem is at fault, because an actual garden could not have a side measuring -7 ft.

WRITTEN EXERCISES

1. If the sum of two consecutive even numbers is 26, and one of them is $\frac{5}{6}$ of the other, find the numbers.

2. If a positive fraction is equal to $\frac{1}{2}$, and the numerator exceeds the denominator by 5, find the fraction.

3. If the sum of two integers is 24, and one of them is $\frac{3}{4}$ of the other, find the numbers.

4. The difference between two numbers is 2, and one of them is 10. Find the other.

5. If there are two numbers such that the first plus 3 times the second equals 8, and twice the first plus 6 times the second equals 15, find the numbers.

EQUATIONS WITH THREE OR MORE UNKNOWNS

316. The definitions and methods given for the solution of two equations with two unknowns may be applied equally well to a greater number of equations and unknowns.

To solve three linear equations with three unknowns, eliminate one unknown from any pair of the equations and the same unknown from any other pair; two equations are thus formed which involve only two unknowns and which may be solved by methods previously given.

Four or more equations with four or more unknowns may be solved similarly.

EXAMPLE

Solve:
$$\begin{cases} x - 2y + 3z = 2, & (1) \\ 2x - 3y + z = 1, & (2) \\ 3x - y + 2z = 9. & (3) \end{cases}$$

To eliminate x from (1) and (2):

Multiplying (1) by 2, $2x - 4y + 6z = 4.$ (4)

Subtracting (2) from (4), $-y + 5z = 3.$ (5)

To eliminate x from (1) and (3):

Multiplying (1) by 3, $3x - 6y + 9z = 6.$ (6)

Subtracting (3) from (6), $-5y + 7z = -3.$ (7)

To eliminate y from (5) and (7):

Multiplying (5) by 5, $-5y + 25z = 15.$ (8)

Subtracting (7) from (8), $18z = 18.$ (9)

$\therefore z = 1.$ (10)

To eliminate z from (8):

Substituting $z = 1$ in (8), $-5y + 25 = 15.$ (11)

Solving (11), $y = 2.$ (12)

Substituting $y = 2$, $z = 1$ in (1), $x - 4 + 3 = 2.$ (13)

Solving (13), $x = 3.$ (14)

TEST. (2) $2 \cdot 3 - 3 \cdot 2 + 1 \cdot 1 = 1.$

(3) $3 \cdot 3 - 1 \cdot 2 + 2 \cdot 1 = 9.$

Since x was found in step (13) by substituting $y = 2$, $z = 1$ in equation (1), it is not necessary to substitute again in this equation when testing the results.

317. Literal equations are solved in the same way; when fractions are involved use the method of Sections 305 and 307.

WRITTEN EXERCISES

Solve:

1. $2x+3y+4z=20,$
$3x+4y+5z=26,$
$3x+5y+6z=31.$

2. $x+y+z=5,$
$x+y-z=7,$
$x-y-z=3.$

3. $x+2y=7,$
$y+2z=2,$
$3x+2y=z-1.$

4. $5x+3y=65,$
$2y-z=11,$
$3x+4z=57.$

5. $y+z=-a,$
$x+z=-b,$
$x+y=-c.$

6. $x+y-z=1,$
$8x+3y-6z=1,$
$4x+y-3z=1.$

7. $x+y+2z=2(b+c),$
$x+2y+z=2(a+c),$
$2x+y+z=2(a+b).$

8. $\frac{1}{2}x+\frac{1}{3}y=12-\frac{1}{6}z,$
$\frac{1}{2}y+\frac{1}{3}z=8+\frac{1}{6}x,$
$\frac{1}{2}x+\frac{1}{3}z=10.$

9. $x+ay=b,$
$ax+z=c,$
$z+cy=a.$

10. $x+\frac{1}{2}y=100,$
$y+\frac{1}{3}z=100,$
$z+\frac{1}{4}x=100.$

11. $7x+13y=205,$
$14x+5z=300,$
$12y+20z=140.$

12. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=31,$
$\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=23.5,$
$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=19.$

13. $x+z=1-cy,$
$y+z=-cx,$
$x+y=-1-cz.$

14. $x=6+\frac{y}{3},$
$y=4+\frac{z}{5},$
$z=8+\frac{x}{4}.$

15. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1,$
$\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=-3,$
$\frac{3}{x}-\frac{4}{y}-\frac{5}{z}=14.$

SUGGESTION. First solve for $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}.$

16. $ax+by=1,$
$cy-az=1,$
$bz-cx=1.$

SUGGESTION. Multiply the equations by c, b, a, respectively, and add.

17. A man has three sums invested, the first at 4 %, the second at 5 %, and the third at 6 % per annum. The annual yield of the first and second sums together is $ 220, that of the first and third together is $ 420, and that of the second and third is $ 400. How many dollars are there in each sum?

18. A tourist spent $ 520 on a trip. If he had cut down his transportation expenses $\frac{1}{3}$, his hotel bill $\frac{1}{4}$, and his miscellaneous expenses $\frac{1}{2}$, his trip would have cost him $ 350. If he had cut down his transportation expenses $\frac{1}{2}$, increased his hotel bills by $\frac{1}{3}$, and his miscellaneous expenses by $\frac{1}{4}$, the trip would have cost him $ 535. Find the amount he actually spent for each of the three items.

19. Find three numbers such that the difference between the reciprocals of the first and second is $\frac{1}{6}$, between the reciprocals of the first and third is $\frac{1}{4}$, and the sum of the reciprocals of the second and third is $\frac{7}{12}$.

20. The sum of the reciprocals of three numbers is $\frac{13}{6}$; the difference between the reciprocals of the first and second equals that between the reciprocals of the second and third. The third number is twice the first. Find the numbers.

21. Three brothers, A, B, C, at a family reunion were discussing their ages. C said to A, "Thirty years ago my age was double yours." Then B said to A, "Twenty-three years ago *my* age was double yours." If C's present age exceeds A's by four years, and B's exceeds A's by eleven years, find the age of each.

318. Equations with More than Three Unknowns. The solution of a particular example will serve to indicate the method to be used when there are more than three unknowns.

EXAMPLE

$$\text{Solve:}\quad \begin{cases} w+x+y+z=8, & (1) \\ 2w-\frac{1}{2}x+\frac{1}{3}y-z=1, & (2) \\ 3w-2x-y+2z=17, & (3) \\ 5w-x-2y-3z=-9. & (4) \end{cases}$$

To eliminate z:

Adding (*1*) and (*2*), $3w + \frac{1}{2}x + \frac{4}{3}y = 9.$ (*5*)

Subtracting twice (*1*) from (*3*), $w - 4x - 3y = 1.$ (*6*)

Adding three times (*1*) and (*4*), $8w + 2x + y = 15.$ (*7*)

To eliminate x:

Subtracting (*7*) from 4 times (*5*), $4w + \frac{13}{3}y = 21.$ (*8*)

Adding twice (*7*) to (*6*), $17w - y = 31.$ (*9*)

To solve for w:

Substituting y from (*9*) into (*8*), $4w + \frac{13}{3}(17w - 31) = 21.$ (*10*)

Solving (*10*), $233w = 466$, and $w = 2.$ (*11*)

To solve for the remaining unknowns:

Substituting $w = 2$ in (*9*), $y = 3.$ (*12*)

Substituting $y = 3$, $w = 2$ in (*7*), $16 + 2x + 3 = 15.$ (*13*)

Solving (*13*), $x = -2.$ (*14*)

Substituting $x = -2$, $y = 3$, $w = 2$ in (*1*), $2 - 2 + 3 + z = 8.$ (*15*)

Solving (*15*), $z = 5.$ (*16*)

Test. (*1*) Used in (*15*),
(*2*) $2 - \frac{1}{2} \cdot (-2) + \frac{1}{3} \cdot 3 - 5 = 1.$
(*3*) $3 \cdot 2 - 2 \cdot (-2) - 3 + 2 \cdot 5 = 17.$
(*4*) $5 \cdot 2 - (-2) - 2 \cdot 3 - 3 \cdot 5 = -9.$

In elementary algebra little emphasis should be placed on problems with more than three unknowns; in later mathematics better and more general methods are given for dealing with them, together with proofs of the assumptions which are tacitly made at this stage.

WRITTEN EXERCISES

Solve:

1. $v + w = 6,$
$2 - w = 8,$
$3z - 2v = 10,$
$2x - y = 12,$
$y + z = 14.$

2. $x - 2y + 32 = 32,$
$2w - 3v + 4x = 13,$
$4y - 2 + 3w = -4,$
$3z - 7w + 5v = 0,$
$5v - x + 3y = -14.$

3. $\frac{1}{8}x + 3y = 23,$
$x + \frac{1}{4}z = 8,$
$y + 3z = 31,$
$x + w + y = 22.$

4. $w + x + y = 9,$
$x + y + z = 9,$
$y + z + w = 9,$
$z + w + x = 9.$

Suggestion. Add the equations, divide by 3, and subtract each from the result.

5. $3w - x + y - 2z = -3,$
$w - 3x - 2y + z = 1\tfrac{1}{2},$
$5w + 2x - 3y + 2z = 19,$
$2w - 2x + 4y - 3z = -15\tfrac{1}{2}.$

6. $w - x + y = 6,$
$x - y + z = -4,$
$y - z + w = 5,$
$z - w + x = -6.$

7. $w + y + z = 2x,$
$w + x + 2 = 3y,$
$w + x + y = 42,$
$w - x - y = w.$

8. $7x - 3y = 1,$
$8z - 7x = 1,$
$4z - 7y = 1,$
$14x - 3w = 1.$

9. $\frac{1}{x} - \frac{1}{y} = \frac{1}{6},$
$\frac{1}{y} - \frac{1}{z} = \frac{1}{12},$
$\frac{1}{x} + \frac{1}{z} = \frac{3}{4}.$

10. $\frac{a}{x} + \frac{b}{y} = m,$
$\frac{b}{y} + \frac{c}{z} = n,$
$\frac{a}{x} + \frac{c}{z} = p.$

11. $x + y + z = 1,$
$5x + y - .22 = .5,$
$2x + 3y + 3z = 1.$

REVIEW

WRITTEN EXERCISES

Solve and test:

1. $x + y = 27,$
$x - y = 17.$

2. $3x + 5y = 19,$
$7x - 4y = 13.$

3. $\tfrac{1}{8}(7 + x) = \tfrac{1}{5}(9 + y),$
$\tfrac{1}{7}(11 + x + y) = \tfrac{1}{6}(9 + y).$

4. $bx + ay = b,$
$ax - by = a.$

5. $x + y + z = 4,$
$2x + 3y - z = 1,$
$3x - y + 2z = 1.$

6. $\tfrac{1}{8}x + \tfrac{1}{4}y = 5,$
$\tfrac{5}{6}x - \tfrac{3}{8}y = 8\tfrac{5}{6}.$

7. $2(3x+1) - 3(4y-26) = 2,$
$3(x - 5) + 2(y - 14) = 6.$

8. $2p + 3w = 1,$
$5p + 7w = 6.$

9. $4h + 7v = 9,$
$2h + 5v = 6.$

10. $2x + 6y + 7z = 4,$
$3x + 8y + 9z = -2,$
$4x + 9y + 10z = 1.$

11. $4x + 11y + 2z = 33,$
$15x + 39y + 7z = 115,$
$23x + 56y + 10z = 162.$

12. $2x - 9y = -1,$
$5x - 24y = 2.$

13. $\frac{x}{a}+\frac{y}{b}=7,$

$\frac{x}{2}+\frac{y}{3}=2a+b.$

14. $3x+5y=1,$

$4x+6y=3.$

15. $\frac{x}{a}+\frac{y}{b}=1,$

$\frac{x}{b}-\frac{y}{a}=1.$

16. $6x+2y+8z=1,$

$9x+8y+11z=-1,$

$15x+12y-3z=-1.$

17. $x-2y+3z-u=5,$

$y-2z+3u-x=0,$

$z-2u+3x-y=0,$

$u-2x+3y-z=0.$

18. Solve for a and b:

$$ac+bq=d,$$
$$ad+5b=q.$$

Solve the equations of Exercise 18 for:

19. a and q.

20. c and d.

21. c and q.

22. b and d.

23. A certain number is written with two digits; twice the tens' digit plus the units' digit makes 9; when the digits are interchanged, the number formed is 27 greater than the given number. Find the original number.

SUGGESTION. 1. Let $x =$ the tens' digit, and $y =$ the units' digit.

2. Then the number is $10x + y$, and $10y + x$ is the number with digits interchanged.

3. Then, by the conditions of the problem,

$$2x + y = 9; \text{ and } 10y + x - 27 = 10x + y.$$

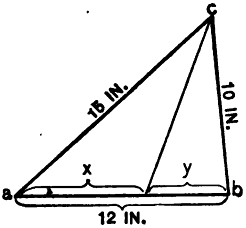
4. Solve these equations for x and y.

24. In a certain number of two digits the sum of three times the tens' digit and twice the units' digit is 36; if the digits are interchanged, the number formed is 27 greater than the original number. What is the original number?

25. The sum of two numbers is 1.6 and their difference is .2. What are the numbers?

26. The difference between two numbers is 30, and the less is $\frac{2}{5}$ of the greater. What are the numbers?

27. Two numbers are such that if the first is increased by 14, the result is twice the second; if the second is diminished by 12, the result is $\frac{1}{3}$ the first. What are the numbers?



28. The angle c in the figure is bisected; and it is known that the bisector of an angle divides the opposite side of the triangle into parts proportional to the adjacent sides; what is the ratio of x to y in the figure? What is the sum of x and y? Find the length of each segment.

29. A person leaves an estate worth $13,000; some of it is willed to a college, and 12 times as much to an eldest son, whose share is $1\frac{1}{2}$ times as much as that of each of his 2 brothers, and $1\frac{2}{3}$ times that of each of 5 sisters. Find the amount left the college.

30. The sum of two numbers is a^2 and their difference is b^2; what are the numbers?

31. A and B play at a game with counters. In the first game A loses as many counters as B has; in the second game B loses as many counters as A then has; at the end of the second game A has 16 counters and B has 4. How many had each at first?

32. A certain number is twice another; their difference divided by their sum equals the smaller. Find the numbers.

33. What fraction becomes equal to $\frac{2}{3}$ if the numerator and the denominator are each increased by 1, and equal to $\frac{1}{2}$ if they are each diminished by 1?

Suggestion. Let $\frac{x}{y}$ be the fraction.

34. A fraction whose value is $\frac{5}{6}$ assumes the value $\frac{7}{8}$ if the numerator and the denominator are each increased by 8. Find the fraction.

35. A fraction becomes equal to $\frac{2}{3}$, if the numerator and the denominator are each increased by 3; and equal to $\frac{4}{7}$, if the numerator and the denominator are each diminished by 3. Find the fraction.

36. The number 128 is the sum of two numbers such that $\frac{1}{7}$ of one equals $\frac{1}{25}$ of the other. What are these numbers?

37. A certain capital is invested in two kinds of securities, one paying 4%, the other $4\frac{1}{2}$%; $\frac{2}{3}$ of the capital is invested in the first kind and the rest in the second; the total income is $75. What is the capital?

38. A certain kind of woolen cloth 1 yd. wide shrinks $\frac{1}{20}$ of its length and $\frac{1}{16}$ of its width in washing. How many yards must be bought in order to have 38 sq. yd. after shrinking?

39. A band of smugglers found a cave, which would exactly hold the cargo of their boat, namely, 13 bales of silk and 33 casks of rum. While they were unloading a revenue cutter was sighted, and they sailed away, leaving 9 casks and 5 bales; these filled only one third of the cave. How many bales alone would the cave hold? How many casks?

Suggestion. Let x equal the number of bales alone required to fill the cave and y the number of casks. Then, 1 bale will fill $\frac{1}{x}$ of the cave and 1 cask will fill $\frac{1}{y}$ of it; also according to the problem $\frac{13}{x}+\frac{33}{y}=$ the whole capacity of the cave, or 1.

Similarly, form the equation corresponding to $\frac{1}{3}$ of the capacity.

40. A man had two sums invested, one at 4 %, the other at 5 %, simple interest, and thus received $500 annually. If the rates of interest had been 5 % and 6 %, respectively, he would have received $110 more per annum. Find the sums invested.

41. A man can walk $2\frac{1}{2}$ miles an hour up hill and $3\frac{1}{2}$ miles an hour down hill. He walks 56 miles in 20 hours on a road no part of which is level. How much of it is up hill?

42. Three thousand dollars was given annually to a college to provide annual scholarships of grades *a*, *b*, *c*. When two of grade *a*, six of grade *b*, and one of grade *c* were granted, the gift was just sufficient; similarly, when four of grade *a*, two of grade *b*, and two of grade *c* were granted, and also when one of grade *a*, five of grade *b*, and five of grade *c* were granted, the gift was sufficient. What was the value of each scholarship?

43. The number of adults and the number of children likely to attend a certain entertainment was estimated in advance; the sum to be raised by the entertainment was $550; if the admission for adults was fixed at 40 cents and that for children at 30 cents, the estimated receipts would lack $90 of the required amount; but if the admission was fixed at 50 cents for adults and 25 cents for children, the exact sum would be raised. How many of each were expected to attend?

44. A tank contains 20 gal. of water, and water flows in at the rate of 5 gal. per minute. At the same time a second tank contains 50 gal. of water, and water flows in at the rate of 2 gal. per minute. Construct a graph to represent the amount of water in the first tank for each minute from 0 to 15. In the same figure draw a graph to represent the amount of water in the second tank for each minute. From the graph read at what time the two tanks will contain equal amounts of water and what the amount is. Verify by solving algebraically.

45. A merchant pays $10 rent weekly. His profits on his sales average 20%. Represent graphically his net profits corresponding to weekly sales ranging from 0 to $200. A second merchant sublets part of his store for $5 per week more than his own rental, but he makes only 10% average profits on his sales. In the same figure represent his total profits for sales ranging from 0 to $200. From the graph read the amount of sales for which both merchants make the same net profit. Verify by solving algebraically.

46. If tin and lead lose, respectively, $\frac{5}{37}$ and $\frac{2}{23}$ of their weights when weighed in water, and a 60-lb. mass of lead and tin loses 7 lb., find the weight of the tin in this mass.

47. What are the sides of a rectangle such that: (*a*) the area is not changed if the base is diminished by 2 and the altitude increased by 2; (*b*) the area is increased by 10, if both base and altitude are increased by 1?

48. The income from a certain investment is devoted to scholarships of two grades; the higher grade receives $200 more than the lower per scholarship. When there are 7 students holding the lower grade and 7 holding the higher, the income exceeds the expense by $100; but the income is exactly sufficient to provide 8 scholarships of the higher grade and 5 of the lower grade. Find the amount of the income, and the amount of each grade of scholarships.

49. The sum of the reciprocals of the first and third of three numbers is twice the reciprocal of the second; the reciprocal of the third is 4 times that of the first; the sum of the reciprocals of the first and second is 7. Find the numbers.

50. A man has three debtors, of whom A and B together owe him 60 pounds, A and C 80 pounds, and B and C 92 pounds. How much did each one owe? (Saunderson's *Algebra*, 1740.)

51. A vessel filled with water has three orifices, A, B, C. If all three are opened, it is emptied in 6 hr.; through B alone it is emptied in $\frac{3}{4}$ of the time that it would take through A alone; and the time through C is twice as great as through B. In what time is the vessel emptied through each orifice alone? (Bossut's *Algebra*, 1773.)

52. The price of a house is 100 dollars. A could pay for it if he had half of B's money in addition to his own; B could pay for it if he had one third of C's; and C could pay for it if he had one fourth of A's money. How much had each? (Euler's *Algebra*, 1770.)

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What are *independent equations?* Sec. 294.

2. What are *simultaneous equations?* Sec. 296.

3. State and explain a method of solving simultaneous equations. Another method. Secs. 299, 301.

4. When should the method of adding be preferred? Sec. 299.

5. When should the method of substitution be preferred? Sec. 301.

6. What is a general form of a system of two simultaneous linear equations? Sec. 302.

7. State the formulas for the values of the unknowns in a system of two simultaneous linear equations. Sec. 303.

8. In solving fractional simultaneous equations, when is it best *not to clear of fractions?* When may the equations be cleared of fractions? Secs. 305, 307.

9. When are equations *dependent?* Sec. 311.

10. When are two equations *inconsistent* or *contradictory?* Sec. 313.

11. What is meant by *interpretation of results?* Sec. 315.

12. State a method for solving a system of three or more simultaneous linear equations of the first degree. Sec. 316.

CHAPTER XX

INVOLUTION AND EVOLUTION

INVOLUTION

319. The operation of raising an expression to a given power is called **involution.**

An important case is the involution of a binomial.

320. PREPARATORY.

1. We have already found that

$$(a+b)^2 = a^2 + 2\,ab + b^2$$
$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

$(a+b)^4$ may be found by multiplying the last result by $a+b$. Thus,

$$\begin{array}{rlllll}
 & a^3 & +\,3\,a^2b & +\,3\,ab^2 & +\,b^3 & \\
 & a & +\,b & & & \\
\hline
 & a^4 & +\,3\,a^3b & +\,3\,a^2b^2 & +\,ab^3 & \\
 & & a^3b & +\,3\,a^2b^2 & +\,3\,ab^3 & +\,b^4 \\
\hline
\therefore\ (a+b)^4 = & a^4 & +\,4\,a^3b & +\,6\,a^2b^2 & +\,4\,ab^3 & +\,b^4
\end{array}$$

2. From this find similarly $(a+b)^5$.

3. From $(a+b)^5$ find similarly $(a+b)^6$.

321. The result of multiplying out a power of a binomial is called a **binomial expansion.**

322. The coefficients of the successive powers of a binomial may be arranged in a triangular table:

EXPANSIONS	COEFFICIENTS
$(a+b)^0 = 1$	1
$(a+b)^1 = a + b$	1 1
$(a+b)^2 = a^2 + 2\,ab + b^2$	1 2 1
$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3$	1 3 3 1
$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$	1 4 6 4 1

Supply the next two lines, using the results of Exercises 2 and 3, Section 320.

323. Binomial Coefficients. The table on the right in Section 322 is called **Pascal's triangle,** and the numbers the **binomial coefficients.**

Each number of Pascal's triangle is the sum of the number directly above it and the number to the left of that.

That this must be so follows from the process of multiplying by $a + b$, and is readily seen when the method of detached coefficients is used (Chapter XXIV). The table enables us easily to write expansions of successive powers of $a + b$.

324. Any expression that can be put into the form of a binomial expansion may be written as a power of a binomial by inspection.

For example:

1. $16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$
$= (2x)^4 - 4 \cdot 8x^3y + 6 \cdot 4x^2y^2 - 4 \cdot 2xy^3 + y^4$
$= (2x)^4 + 4 \cdot (2x)^3(-y) + 6(2x)^2(-y)^2 + 4(2x)(-y)^3 + (-y)^4$
$= (2x - y)^4$.

2. $a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3$
$= (a + b)^3 + 3(a + b)^2(-c) + 3(a + b)(-c)^2 + (-c)^3$
$= (\overline{a + b} - c)^3$.

ORAL EXERCISES

Express as a power of a binomial:

1. $-a^3 + 3a^2 - 3a + 1$.
2. $a^{12} + 3a^8 + 3a^4 + 1$.
3. $1 - 3x^4 + 3x^8 - x^{12}$.
4. $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$.
5. $8x^3 - 12a^2b + 6ab^2 - b^3$.
6. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.
7. $x^4 - 4x^3 + 6x^2 - 4x + 1$.
8. $(2x)^4 - 4(2x)^3 + 6(2x)^2 - 4(2x) + 1$.
9. $16x^4 + 32x^3 + 24x^2 + 8x + 1$.
10. $16x^4 - 32x^3 + 24x^2 - 8x + 1$.
11. $a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1$.
12. $a^2 + 2ab + b^2 - 2ac - 2bc + c^2$.

13. $(2a)^4 + 4(2a)^3 + 6(2a)^2 + 4(2a) + 1.$

14. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$

15. $(a-b)^4 - 4(a-b)^3c + 6(a-b)^2c^2 - 4(a-b)c^3 + c^4.$

16. $x^6 - 6x^5(2y) + 15x^4(2y)^2 - 20x^3(2y)^3 + 15x^2(2y)^4 - 6x(2y)^5 + (2y)^6.$

17. State in order the coefficients in the expansion of a binomial of the fourth degree. Of the third degree. Of the fifth degree.

325. The Binomial Formula. It can be proved that

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2 \cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4}a^{n-4}b^4 + \cdots.$$

This is known as the **binomial formula.**

The factor 1 is understood in each denominator; and by denoting the product $1 \cdot 2 \cdot 3$ by $3!$ (read "three factorial"), and generally $1 \cdot 2 \cdot 3 \cdots k$ by $k!$, the above formula can be written:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 + \cdots.$$

326. The expansion of $(a+b)^n$ may be written by observing that:

1. *The first term* a^n, *the last is* b^n, *and the number of terms is* $n+1$.

2. *The exponent of* a *is one less in each succeeding term.* b *occurs in the second term, and its exponent increases by one in each succeeding term.*

3. *The coefficient of any term is the coefficient of the next preceding term multiplied by the exponent of* a *in that term, and divided by one more than the exponent of* b.

4. *All of the signs are plus if the sign of* b *is plus, or alternately plus and minus if the sign of* b *is minus.*

WRITTEN EXERCISES

1. Test the binomial formula just given for $n = 3$.

It should reduce to the known expression for $(a + b)^3$. The formula comes to an end because the factor $n - 3$ occurs in all the terms after a certain one, and when $n = 3$, this factor is zero.

2. Test similarly for $n = 4, 5, 6, 2, 1$.

Expand:

3. $(x + y)^4$.
4. $(x - y)^4$.
5. $(x - y)^5$.
6. $(x + y)^5$.
7. $(x - 1)^5$.
8. $(x - 2y)^6$.
9. $(ab + 1)^4$.
10. $(bc - 1)^4$.
11. $(ab - cd)^4$.
12. $(t - u)^6$.
13. $(3a + b)^7$.
14. $(x + 5)^5$.

15. Write the next two terms of the binomial formula as given in Sec. 325.

16. Observe that if the successive terms were written according to the same law, the *tenth* term of the binomial formula would be $\frac{n(n-1)(n-2)\cdots(n-8)}{9!}a^{n-9}b^9$. Write the fifteenth term.

17. By reference to the binomial formula, state the number of the last term written in the following expression:

$$(x+2y)^8 = x^8 + 8x^7(2y) + \frac{8 \cdot 7}{2!}x^6(2y)^2 + \cdots + \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{6!}x^2(2y)^6 + \cdots.$$

Write the first three terms and the seventh term in the expansion of each of the following:

18. $(x + y)^{12}$.
19. $(1 + 5x)^{14}$.
20. $(ab - y)^{11}$.
21. $\left(\frac{x}{2} + \frac{y}{3}\right)^{12}$.
22. $(2x - 1)^9$.
23. $(4x + 3)^{20}$.

24. Reduce the terms written for Exercise 21 to their simplest form.

25. If the expansions in Exercises 18–23 were written out in full, how many terms would each have?

26. In each of Exercises 18–23, determine whether or not there is a middle term in the expansion. If there is a middle term:

(*a*) Determine its number.
(*b*) Write it out, without simplifying.
(*c*) Simplify the result.

Expand:

27. $\left(\frac{1}{a^2} - \frac{a}{3}\right)^5$.

28. $(a^2b - \frac{2}{3})^7$.

29. $\left(\frac{1}{x} - \frac{2}{y}\right)^5$.

30. $(x^p - 5y^q)^5$.

31. Write the first five terms of $\left(y^2 - \frac{b}{2y}\right)^8$.

32. Write the first five terms of $\left(2x - \frac{y^2}{3}\right)^6$.

33. Write the first four terms of $(2x^2 - y^3)^8$.

Write the last four terms of:

34. $(x^2 - \frac{1}{3})^9$.

35. $\left(x^2 - \frac{y}{2}\right)^6$.

EVOLUTION

327. The process of extracting an indicated root is called **evolution.** The most important case of evolution is the extraction of square root.

All the numbers that we have hitherto considered, whether positive or negative, have positive squares; none of them has a negative square, consequently the square root of a negative number (as, for example, $\sqrt{-5}$) has no meaning at this stage of our work, but will be explained in Chapter XXVIII.

328. Square Root by Inspection. The formula

$$[\pm(a + b)]^2 = a^2 + 2ab + b^2.$$

shows that when one of the terms of the trinomial is twice the product of the square roots of the other two, the trinomial is the square of the sum of these square roots. By aid of this relation the square roots of certain trinomials can be found readily by inspection.

For example:

$\sqrt{4a^2 - 12ab + 9b^2} = \pm(2a - 3b)$, since $-12ab$ is twice the product of the square roots of $4a^2$ and $9b^2$. These roots must be taken with opposite signs in this case because twice their product is to be negative.

ORAL EXERCISES

Find the square root of:

1. $x^2 + 4x + 4$.
2. $4x^2 + 8x + 4$.
3. $a^2b^2 + 2ab + 1$.
4. $a^2 - 4ab + 4b^2$.
5. $4m^2 - 4mn + n^2$.
6. $x^4 - 4x^2 + 4$.
7. $16x^8 + 8x^4 + 1$.
8. $4a^2b^2 - 4ab + 1$.
9. $a^6 + 2 + \frac{1}{a^6}$.
10. $x^{4a} + 2x^{2a}y^b + y^{2b}$.
11. $25a^4 - 10a^2 + 1$.
12. $25a^2 - 30ab + 9b^2$.
13. $49a^2b^2 - 14a^3b + a^4$.
14. $25a^4b^2c^2 + 10a^2bc^5 + c^8$.

329. Square Roots of Arithmetical Numbers. The square root of arithmetical numbers can be found approximately by inspection.

EXAMPLES

1. Find approximately $\sqrt{19}$.

19 lies between 16 and 25.

Therefore $\sqrt{19}$ lies between $\sqrt{16}$ and $\sqrt{25}$, or between 4 and 5.

That is, $\sqrt{19}$ is 4 plus a decimal.

2. Find approximately $\sqrt{643}$.

643 lies between 400 and 900.

Therefore, $\sqrt{643}$ lies between $\sqrt{400}$ and $\sqrt{900}$, or between 20 and 30.

That is, it is 20 plus a number less than 10.

The numbers to be added in any case will not change the first figure of the root found. That is, *by inspection we can find exactly the first figure of the square root.*

330. Pointing off into Periods. Since $10^2 = 100$, we know that the square root of any number greater than 1 but less than 100 is less than 10. Its integral part consists of one figure.

Since $100^2 = 10{,}000$, we know that the square root of any number greater than 100 but less than 10,000 is greater than 10 but less than 100.

That is, if the given number has 3 or 4 digits in its integral part, its square root will have 2 digits in its integral part. If larger numbers are given, the above reasoning can be repeated for 1000^2, etc., showing that in all cases if the number be pointed off into periods of 2 digits each (or possibly fewer in the left period), then each period will correspond to a digit of the root.

Thus in 67′62′31, there are three periods, hence there are three places in the integral part of the root. Since 67 lies between 64 and 81, the square root of 67 is approximately 8, and of 676,231 approximately 800.

ORAL EXERCISES

By the method above state an approximate square root of:

1. 12′36.
2. 30′95.
3. 1′25′00.
4. 8′23′00.
5. 43′21′00.
6. 58′61′23.

331. When the first digit of the square root has been found by inspection, the process may be continued thus:

EXAMPLE

Find $\sqrt{2209}$:

1. The approximate value, as above, is 40.
2. Let $\sqrt{2209} = 40 + r$, where r is the rest of the root.
3. $\therefore 2209 = (40 + r)^2 = 40^2 + 2 \cdot 40 \cdot r + r^2$.
4. $\therefore 2209 - 40^2 = 2 \cdot 40 \cdot r + r^2$, or $609 = 80\,r + r^2$.
5. Then 609 is greater than $80\,r$, or $\frac{609}{80} = 7 +$ dec. is greater than r.

$\therefore 7 +$ decimal is greater than r, and it is possible that $7 = r$.

Trying, we find that $80 \cdot 7 + 7^2 = 609$, hence, $r = 7$.

Therefore, $\sqrt{2209} = 40 + 7 = 47$.

332. Thus, when once an approximate value, a, has been found for the root, an approximate value for the remainder, r, of the root can be found by means of the formula: $(a+r)^2 = a^2 + 2\,ar + r^2$.

1. Let n denote the number whose square root is sought, a denote the approximate root at any stage, and r the remainder of the root.
2. Then, $n = (a + r)^2 = a^2 + 2\,ar + r^2$.
3. $\therefore n - a^2 = 2\,ar + r^2$.
4. Or, $n - a^2$ is greater than $2\,ar$, or, $\frac{n - a^2}{2\,a}$ is greater than r.
5. Hence, $\frac{n - a^2}{2\,a}$ may be tried as an approximate value of r.

333. What precedes may be formulated into a process or working rule, thus:

1. *Point off the number into periods of two figures each, beginning at units' place (at the decimal point).*

2. *By inspection find the largest integer whose square is not greater than the left period.* (In Example A it is 9.)

3. *Use this integer as the first digit of the root. Subtract its square from the left period.* (In Example A this square is 81.)

4. *Bring down the next period.* (In Example A this makes 364.)

5. *Multiply the part of the root already found by 2. This number is called the* **trial divisor.** (18 in Example A.)

(A)

Root	9 2
Number	84'64
	81
18	364
182	364

6. *Divide the remainder (omitting the right digit) by the trial divisor and use the digit found as the next digit of the root.* (In Example A, $36 \div 18 = 2$.)

7. *Annex this digit to the trial divisor. This forms the* **complete divisor**. (182 in A.)

8. *Multiply the complete divisor by the digit of the root just found and subtract.*

Note. It may happen that the product to be subtracted is larger than the number from which it is to be subtracted. This indicates that the trial divisor produced too large a digit. Try the next smaller digit for the figure of the root last found.

(B)

Root	3 0. 6 9
Number	9'41'.87'61
	9
6	41
60	4187
606	3636
612	55161
6129	55161

9. *Repeat the steps* 4 *to* 8 *until all of the periods have been brought down.*

If the last remainder is zero, as in Example B, the process is ended, the given number is a perfect square, and its root has been found exactly. If the last remainder is not zero, as in C, the process may be continued as far as desired by supplying zeros.

Test. The square of the root, if complete, equals the given number.

(C)

Root	1 4. 1 4+
Number	2'00.
	1
2	100
24	96
28	400
281	281
282	11900
2824	11296
	604

334. When a number contains a decimal the decimal point of its root is placed between the figures furnished by the integral periods and those furnished by the decimal periods (as in C, Sec. 333).

WRITTEN EXERCISES

Find the square roots:

1. 361.
2. 784.
3. 625.
4. 841.
5. 2025.
6. 1936.
7. 177,241.
8. 120,409.
9. 4,334,724.
10. 4,888,521.

Find the square roots to two decimal places:

11. 2.25.
12. 7.84.
13. 6.25.
14. 19.36.
15. 90.25.
16. 1.21.
17. 2000.
18. 0.03.
19. 5.
20. 3.
21. 111.
22. 0.00111.

335. To find the square roots of fractional numbers, *either first reduce the fraction to a decimal or extract the square root of both numerator and denominator.*

WRITTEN EXERCISES

Find the square roots:

1. $\frac{81}{144}$.
2. $\frac{169}{225}$.
3. $\frac{256}{441}$.
4. $\frac{361}{841}$.
5. $\frac{625}{3025}$.
6. $\frac{289}{961}$.
7. $\frac{622521}{724201}$.
8. $\frac{7695076}{7871225}$.

9. It is known that if the sides of a rectangular prism are a, b, and c, the diagonal, d, is $\sqrt{a^2+b^2+c^2}$. If the sides of such a solid are 20 yd., 30 yd., and 50 yd., find its diagonal to three decimal places.

10. Compute $\sqrt{s(s-a)(s-b)(s-c)}$ to two decimal places, when $s=18.5$ in., $a=10$ in., $b=15$ in., $c=12$ in.

Solve and compute the roots to three decimal places:

11. $9x^2=21$.
12. $16.1x^2=21$.

336. Square Roots of Polynomials. The square root of every polynomial that is a square may be extracted according to the process given in Sec. 333. If the polynomial is not a square, the square root may be approximated to any number of terms.

EXAMPLE

Extract the square root of $a^2 - 2ab + b^2 - 2ac + 2bc + c^2$.

$$\begin{array}{ll}
\textbf{Root} & \underline{\pm(a-b-c)\qquad\qquad\qquad\qquad} \\
\textbf{Power} & a^2 - 2ab - 2ac + b^2 + 2bc + c^2 \\
 & \underline{a^2 - 2ab \qquad\quad + b^2} \\
 & \qquad\qquad -2ac \qquad + 2bc + c^2 \\
 & \qquad\qquad \underline{-2ac \qquad + 2bc + c^2}
\end{array}$$

1. As far as possible, arrange the terms according to the descending powers of some letter, as a in this case.

2. The square root of the first term is the first term of the root. (Corresponds to steps 2, 3, Sec. 333.)

3. Divide the second term of the power by twice the first term of the root, $2a$ in this case. The result is the second term of the root. (Steps 5, 6.)

4. Subtract from the power the square of the binomial found.

5. If there is a remainder (as $-2ac + 2bc + c^2$ in this case) it shows that the power contains the square of a trinomial and that there is at least another term in the root.

This term (c) is found by dividing the remainder by twice the part of the root found [$2(a-b)$ in this case], for the same reason as in the square root of numbers. (Step 6.)

6. The square of the entire root so far found $(a-b-c)^2$ must now be subtracted. We have already subtracted the square of the first part of the new binomial [$(a-b)^2$ in this case]. Therefore, subtract the rest of the square [$-2(a-b)c + c^2$, or $-2ac + 2bc + c^2$].

Briefly, the trial divisor, $2(a-b)$, is augmented by the next term, $-c$, resulting in $2(a-b)-c$, and this is multiplied by $-c$. This gives the part $-2(a-b)c + c^2$, still to be subtracted. This is analogous to what is done in extracting the square roots of numbers. (Steps 7, 8.)

If there is a new remainder, divide it by twice the entire part of the root found and proceed as before. Test by squaring the root.

WRITTEN EXERCISES

Extract the square root of:

1. $49a^2b^2 - 14a^3b + a^4$.
2. $16x^2y^2 + 40xy^2z + 25y^2z^2$.
3. $4x^4 + 4x^3 + 5x^2 + 2x + 1$.
4. $x^4 - 6x^3 + 11x^2 - 6x + 1$.
5. $1 + 4x + 10x^2 + 12x^3 + 9x^4$.
6. $9x^4 + 12x^3 + 22x^2 + 12x + 9$.
7. $9a^2 + 12ab + 4b^2 + 6ac + 4bc + c^2$.
8. $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.
9. $x^6 - 4x^5 + 6x^4 + 2x^3 - 11x^2 + 6x + 9$.

10. $\frac{4}{x^2}+\frac{9}{y^2}+\frac{25}{z^2}-\frac{12}{xy}-\frac{30}{yz}+\frac{20}{xz}$.

11. $4\,ac-\frac{4\,c}{b}+a^2-\frac{2\,a}{b}+4\,c^2+\frac{1}{b^2}$.

12. $\frac{1}{4}\,x^4-2\,x^3-x+4+\frac{4}{x}+\frac{1}{x^2}$.

13. $2\,xw-4\,xz^3+x^2+w^2+4\,z^6-4\,wz^3$.

14. $a^2+9\,b^2-6\,ab-6\,bc+2\,ca+c^2$.

15. $\frac{1}{4}\,a^2-\frac{1}{3}\,ab+\frac{1}{9}\,b^2+\frac{1}{5}\,ac-\frac{2}{15}\,bc+\frac{1}{25}\,c^2$.

16. $1-12\,y+38\,y^2-12\,y^3+y^4$.

17. $2\,x^2z^2-2\,x^2y-2\,yz^2+x^4+y^2+z^4$.

18. $a^4+6\,a^3+\frac{29}{3}\,a^2+2\,a+\frac{1}{9}$.

19. $x^4y^4-2\,x^2y^2z^2+4\,x^3y^3z+9\,z^4-12\,xyz^3$.

20. $9\,a^2+25\,b^2+9\,c^2-30\,ab+18\,ac-30\,bc$.

21. $m^2n^2+p^2q^2+r^2s^2+2\,mnpq+2\,mnrs+2\,pqrs$.

22. $m^4+q^4+m^2p^2+2\,m^3p-2\,m^2q^2-2\,mpq^2$.

23. $4+29\,a^2-12\,a-30\,a^3+25\,a^4$.

24. $\frac{1}{4}\,x^4-\frac{4}{3}\,x^3+\frac{1}{3}\,x^2y+4\,x^2-2\,xy+\frac{1}{4}\,y^2$.

Extract the square root to 4 terms:

25. a^2-1. 27. y^2+1. 29. $4-z$. 31. $16\,a^2+12\,ab$.
26. $1-x$. 28. x^2+5. 30. $x^2+4\,y$. 32. $9\,m^2+9\,mn$.

REVIEW

WRITTEN EXERCISES

Extract the square root of:

1. 14,641. 2. 1.5625. 3. $a^4+2\,a^2x+x^2$.

4. $9\,a^4+12\,a^3-20\,a^2-16\,a+16$.

5. $a^2+b^2+4\,c^2-2\,ab+4\,ac-4\,bc$.

6. $m^4+4\,am^3+6\,a^2m^2+4\,a^3m+a^4$.

7. $\frac{y^4}{16}-\frac{y^3}{4\,z}+\frac{3\,y^2}{20\,z^2}+\frac{y}{5\,z^3}+\frac{1}{25\,z^4}$.

8. $x^6-6\,x^5+15\,x^4-20\,x^3+15\,x^2-6\,x+1$.

Expand by the binomial formula:

9. $(a+\frac{1}{2})^7$.

10. $(ax+1)^5$.

11. $(mx^2-1)^6$.

12. $(a^3+5)^4$.

13. $(ax^2-2y)^7$.

14. $(3x^2-\frac{1}{2})^5$.

15. $\left(\frac{a}{b}-\frac{b}{a}\right)^4$.

16. $\left(\frac{1}{2x^2}-\frac{1}{3y^2}\right)^5$.

17. Find the middle term of the expansion of $\left(\frac{a}{x}+\frac{x}{a}\right)^{10}$.

18. Raise 98 to the 5th power by the binomial theorem.

SUGGESTION. Use 100 – 2 for 98.

19. Find the ratio between the 6th term in the expansion of $\left(\frac{1+3x}{2}\right)^{10}$ and the 5th term in the expansion of $\left(\frac{1+3x}{2}\right)^9$.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What is *involution?* Sec. 319.
2. What is a *binomial expansion?* Sec. 321.
3. How else can an expression in the form of a binomial expansion be stated? Sec. 324.
4. State the rule for writing a binomial expansion. Sec. 326.
5. Define *evolution.* Sec. 327.
6. On what formula is the general process of finding *square root* based? Sec. 332.
7. How may the square root of a fraction be found? Sec. 335.

HISTORICAL NOTE

The most important process in involution is the binomial expansion. It is the basis for finding the powers of all algebraic polynomials, and likewise the basis of determining roots. The general principle for writing the terms of this series, known as the Binomial Theorem, was discovered by Sir Isaac Newton, the greatest English mathematician of the seventeenth century. Special cases of this formula, like $(a+b)^2$ and $(a+b)^3$, were known to the Hindoos and Arabs, who used them to find square and cube roots of numbers, Vieta in the sixteenth century knew the expansion of $(a + b)^4$, and Pascal constructed a table for mechanically computing the numerical coefficients; but the full significance of the Binomial Theorem was first discovered by Newton while investigating an expression for the value of π, the ratio of the circumference to the diameter of a circle.

Newton was born in a small town, Lincolnshire, in 1642, and his health was so delicate in his childhood that it interfered with his education. It is said that his backwardness provoked the ridicule of his companions, under the sting of which he soon surpassed all of them in learning. He entered Cambridge University in 1660 and readily mastered the works of Descartes, Vieta, and Wallis. While developing methods for finding areas bounded by various curved lines, he discovered, not only the Binomial Theorem, but also the method now known as Calculus, for which Newton will ever be famous as a mathematician. His discovery of the law of falling bodies and the general principle of gravitation has placed his name for all time among the foremost men of science. The English writer Pope has paid a graceful tribute to Newton's greatness in the following couplet:

SIR ISAAC NEWTON

"Nature and nature's laws lay hid in night,
God said, 'let Newton be' and all was light."

But we should not conclude from this poetic conception that Newton's great achievements were mere flashes of thought. They were the product of a mind trained by labor and informed by painstaking study. Genius pointed out the way to greater things, but not till Newton had climbed the height of other men's knowledge.

CHAPTER XXI

RADICALS AND EXPONENTS

DEFINITIONS AND PROPERTIES

337. Rational Numbers. Integers and other numbers expressible as the quotient of two integers are called **rational numbers**

Thus, 5 and $\frac{2}{3}$ are rational numbers.

Also, .2, which is expressible as $\frac{2}{10}$, is a rational number.

338. Irrational Numbers. Any number not rational is called an **irrational number.**

Thus, $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{10}$, $\frac{1}{\sqrt{5}}$, $1 + \sqrt{3}$, $\sqrt{2} - \sqrt{3}$, are irrational numbers.

Numbers like $\pi = 3.14159^+$, the ratio of the circumference to the diameter of a circle, are also called irrationals.

A special class of irrational numbers, namely, even roots of negative numbers are called imaginaries. We have seen that $(+2)(+2) = 4$, and that $(-2)(-2) = 4$; hence, $\sqrt{4} = \pm 2$, similarly $\sqrt{3} = \pm 1.732^+$; but we have not yet found two equal numbers whose product is -4, or any other negative number. Hence, the use of numbers like $\sqrt{-4}$ and $\sqrt{-a}$, will not appear in the processes or in the roots of equations until Chapter XXVIII has been studied.

339. An indicated root of any number is called a **radical.**

Thus, $\sqrt{5}$, $\sqrt[3]{8}$, $\sqrt{\frac{a}{3b}}$, $\sqrt{a + x^2}$, are radicals.

In the present chapter all roots that cannot be exactly extracted by inspection are indicated.

340. Although all indicated even roots may be taken either + or −, it is customary in the treatment of radicals to omit these signs, regarding the radical as positive.

341. Surd. A monomial containing an indicated root of a rational number is sometimes called a **surd.** The part under the radical is called the **radicand.**

342. The number denoting the root to be taken is called the **index** of the root.

Thus, $\sqrt{2}$, $\sqrt[3]{5}$, $3\sqrt[5]{7}$, $5\sqrt{14-a}$, are surds the radicands in order are 2, 5, 7, $14-a$, and the indices of the roots are 2, 3, 5, 2.

343. A radical expression with no rational factor is called an **entire surd**; otherwise a **mixed surd.**

344. The index of the root is called the **order** of a surd.

For example, $\sqrt{2}$ is of the second order, $\sqrt[3]{2}$ is of the third order, $\sqrt[5]{3}$ is of the fifth order.

345. A surd of the second order is also called a **quadratic** surd, of the third order a **cubic** surd, and one of the fourth order a **biquadratic** surd.

346. An expression involving one or more radicals is called a **radical expression.**

Thus, $5+2\sqrt{3}$, $\frac{4}{\sqrt{x}}-1$, $\frac{8+\sqrt{4a}}{2-\sqrt{3b}}$ are radical expressions.

347. Some Properties of Radicals. A few important properties of radicals are given here. The fuller treatment is contained in Chapter XXVI on Exponents.

348. I. $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$.

For example, $\sqrt{2}\cdot\sqrt{3}=\sqrt{6}$.

That this is true may be seen by squaring both members.

Thus, $$(\sqrt{2}\cdot\sqrt{3})(\sqrt{2}\cdot\sqrt{3})=\sqrt{6}\cdot\sqrt{6},$$

or, $$\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{3}=\sqrt{6}\cdot\sqrt{6},$$

or, $2\cdot 3=6$, which is known to be true.

In the same way, it may be seen that for every a and b, $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$.

In words:

The product of two square roots is the square root of the product of the numbers.

WRITTEN EXERCISES

Show by squaring that:

1. $\sqrt{3}\cdot\sqrt{5}=\sqrt{15}$.
2. $\sqrt{4}\cdot\sqrt{7}=\sqrt{28}$
3. $\sqrt{3}\cdot\sqrt{7}=\sqrt{21}$.
4. $\sqrt{5}\cdot\sqrt{11}=\sqrt{55}$.
5. $\sqrt{2a}\cdot\sqrt{3b}=\sqrt{6ab}$.
6. $\sqrt{x^3}\cdot\sqrt{5y}=\sqrt{5x^3y}$.
7. $\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{5}=\sqrt{30}$.
8. $\sqrt{a}\cdot\sqrt{b}\cdot\sqrt{c}=\sqrt{abc}$.

349. II. $\sqrt{a^2b}=\sqrt{a^2}\ \sqrt{b}=a\sqrt{b}$.

In words:

Factors which are perfect squares may be taken from under the radical sign.

Thus, $\sqrt{18}=\sqrt{9}\cdot\sqrt{2}=\sqrt{3^2}\cdot\sqrt{2}=3\sqrt{2}$.

WRITTEN EXERCISES

Take all factors which are perfect squares from under the radical sign:

1. $\sqrt{20}$.
2. $\sqrt{27}$.
3. $\sqrt{50}$.
4. $\sqrt{48}$.
5. $\sqrt{45}$.
6. $\sqrt{75}$.
7. $\sqrt{24}$.
8. $\sqrt{32}$.
9. $\sqrt{12}$.
10. $\sqrt{40}$.
11. $\sqrt{500}$.
12. $\sqrt{128}$.
13. $\sqrt{8a^2}$.
14. $\sqrt{x^3}$.
15. $\sqrt{48x^3y^2}$.
16. $\sqrt{45a^7y^4}$.

350. III. $a\sqrt{b}=\sqrt{a^2}\cdot\sqrt{b}=\sqrt{a^2b}$.

In words:

Any factor outside the radical sign may be placed under the radical sign provided the factor is squared.

Thus, $3\sqrt{2}=\sqrt{9}\cdot\sqrt{2}=\sqrt{18}$.

WRITTEN EXERCISES

Place under one radical sign:

1. $6\sqrt{2}$.
2. $5\sqrt{3}$.
3. $2\cdot\sqrt{3}$.
4. $5\cdot\sqrt{7}$.
5. $3\cdot\sqrt{7}\cdot 2$.
6. $5\cdot\sqrt{3}\cdot\sqrt{2}$.
7. $2\cdot\sqrt{3}\cdot\sqrt{11}$.
8. $5\cdot\sqrt{3}\cdot\sqrt{7}$.
9. $4\sqrt{2}\cdot 3$.
10. $b\sqrt{2}$.
11. $2x\sqrt{3x}$.
12. $ab\sqrt{bc}$.
13. $t\sqrt{g}$.
14. $r\sqrt{\pi r}$.
15. $\frac{x}{3}\sqrt{18xy}$.
16. $a\sqrt{b-a}$.

351. Fractional Exponents. A more convenient notation to indicate roots is found in **fractional exponents.** In order to find the proper meaning of such exponents, we shall assume that the laws of integral exponents apply also to fractional exponents.

Assume the law $a^m \cdot a^n = a^{m+n}$ to hold when $m = \frac{1}{2}$ and $n = \frac{1}{2}$.

Then, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$.

Therefore, $a^{\frac{1}{2}}$ is one of the two equal factors of a, or the *square root of* a.

(1) Hence we may write $\sqrt{a} = a^{\frac{1}{2}}$.

Similarly, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$.

Hence, $a^{\frac{1}{3}}$ is the *cube root of* a.

(2) and $\sqrt[3]{a} = a^{\frac{1}{3}}$.

352. *In general* $\sqrt[n]{a} = a^{\frac{1}{n}}$.

EXAMPLES

1. $9^{\frac{1}{2}} = \sqrt{9} = 3.$
2. $(4\,a^2)^{\frac{1}{2}} = \sqrt{4\,a^2} = 2\,a.$
3. $27^{\frac{1}{3}} = \sqrt[3]{27} = 3.$
4. $(8\,a^3b^6)^{\frac{1}{3}} = \sqrt[3]{8\,a^3b^6} = 2\,ab^2.$

WRITTEN EXERCISES

Write with radical sign and simplify:

1. $4^{\frac{1}{2}}$. 3. $(9\,x^2)^{\frac{1}{2}}$. 5. $(4\,y^2)^{\frac{1}{2}}$. 7. $8^{\frac{1}{3}}$. 9. $(x^3y^3)^{\frac{1}{3}}$.
2. $9^{\frac{1}{2}}$. 4. $(x^2y^2)^{\frac{1}{2}}$. 6. $(25\,x^2)^{\frac{1}{2}}$. 8. $(27\,a^3)^{\frac{1}{3}}$. 10. $(27\,b^6)^{\frac{1}{3}}$.

353. Applying the law of multiplication to $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}}$.

we have $a^{\frac{1}{3}+\frac{1}{3}} = a^{\frac{2}{3}}$.

But, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = (a^{\frac{1}{3}})^2$.

Therefore, $a^{\frac{2}{3}} = (a^{\frac{1}{3}})^2 = (\sqrt[3]{a})^2$.

Similarly, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = (a^{\frac{1}{2}})^3 = (\sqrt{a})^3$.

354. *In general the numerator of the fractional exponent denotes the power to be taken, and the denominator denotes the root.*

In symbols, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$.

EXAMPLES

1. $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4.$

2. $16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = 2^3 = 8.$

3. $9^{\frac{3}{2}} = (\sqrt{9})^3 = 27.$

4. $(4\,a^2b^4)^{\frac{5}{2}} = (2\,ab^2)^5 = 32\,a^5b^{10}.$

WRITTEN EXERCISES

Write with radical sign:

1. $a^{\frac{2}{3}}.$
2. $b^{\frac{4}{3}}.$
3. $c^{\frac{3}{4}}.$
4. $x^{\frac{3}{2}}.$
5. $(2\,a)^{\frac{1}{3}}.$
6. $(xy)^{\frac{3}{2}}.$
7. $3\,b^{\frac{1}{3}}.$
8. $5\,mx^{\frac{2}{3}}.$
9. $a^{\frac{2}{3}}b^{\frac{2}{3}}.$
10. $5\,x^{\frac{1}{3}}(y+z)^{\frac{2}{3}}.$
11. $2\,r^{\frac{5}{4}}s^{\frac{3}{4}}.$
12. $5\,a^{\frac{1}{x}}b^{\frac{m}{x}}.$

Find the rational values of:

13. $16^{\frac{1}{2}}.$
14. $27^{\frac{1}{3}}.$
15. $16^{\frac{1}{4}}.$
16. $27^{\frac{2}{3}}.$
17. $9^{\frac{3}{2}}.$
18. $8^{\frac{4}{3}}.$
19. $64^{\frac{2}{3}}.$
20. $32^{\frac{1}{5}}.$
21. $(-27)^{\frac{2}{3}}.$
22. $(-125)^{\frac{1}{3}}.$
23. $(\frac{1}{4})^{\frac{1}{2}}.$
24. $(\frac{1}{8})^{\frac{1}{3}}.$
25. $(\frac{4}{9})^{\frac{1}{2}}.$
26. $(-\frac{8}{27})^{\frac{1}{3}}.$
27. $(\frac{4}{9})^{\frac{3}{2}}.$
28. $9^{\frac{1}{2}}(\frac{1}{9})^{\frac{1}{2}}.$
29. $16^{\frac{1}{2}}(\frac{1}{4})^{\frac{1}{2}}.$
30. $27^{\frac{2}{3}}(\frac{4}{9})^{\frac{1}{2}}.$
31. $\sqrt[3]{64}\cdot(\frac{1}{16})^{\frac{3}{2}}.$
32. $(-125)^{\frac{2}{3}}\sqrt{\frac{1}{25}}.$
33. $(-343)^{\frac{2}{3}}\sqrt{\frac{1}{49}}.$

Write with fractional exponents and simplify when possible:

34. $\sqrt{x}.$
35. $\sqrt{2\,x}.$
36. $3\sqrt{x}.$
37. $\sqrt{a^3}.$
38. $\sqrt{4\,a}.$
39. $5\sqrt{25\,b^2}.$
40. $3\sqrt[3]{4\,m^2}.$
41. $5\sqrt[3]{8\,z^4}.$
42. $2\sqrt[3]{27\,xy^3}.$
43. $3\sqrt[4]{a^2b^3}.$
44. $6\sqrt[5]{ab^6}.$
45. $-2\,a\sqrt[5]{32\,a^3}.$
46. $5\,a^2\sqrt[3]{3\,a^3}.$
47. $x\sqrt{(yz)^3}.$
48. $ab\sqrt{(a+b)^3}.$
49. $2\sqrt{p^3}\cdot\sqrt[3]{q^2}.$
50. $\sqrt[n]{a^r}\cdot\sqrt[n]{b^{2n}}.$
51. $\sqrt[r]{a^x}\cdot\sqrt[r]{a^{5x}}.$

PROCESSES

355. Reduction. The reduction of radicals means the changing from one form to another equivalent form, not necessarily to a simpler one.

1. To reduce a mixed surd to an entire surd.

EXAMPLES

1. $3\sqrt{5} = \sqrt{3^2 \cdot 5} = \sqrt{45}.$ **Sec. 354.**

2. $2\sqrt[3]{3a} = \sqrt[3]{2^3 \cdot 3a} = \sqrt[3]{24a}.$

To reduce a mixed surd to an entire surd raise the rational factor to a power equal to the index of the root and place the result under the radical sign as a factor.

WRITTEN EXERCISES

Reduce to an entire surd:

1. $\frac{1}{2}\sqrt{2}.$
2. $\frac{1}{a}\sqrt{a}.$
3. $\frac{2}{3}\sqrt{x}.$
4. $2\sqrt[3]{2}.$
5. $3\sqrt[3]{5}.$
6. $a\sqrt[3]{a}.$
7. $a\sqrt{ab}.$
8. $\frac{2}{3}\sqrt{\frac{15}{7}x^2}.$
9. $\frac{4}{5}\sqrt{\frac{7}{a}}.$
10. $x^2\sqrt[3]{y}.$
11. $2\sqrt[4]{3}$
12. $\frac{1}{3}\sqrt[4]{3\frac{3}{4}}.$
13. $\frac{1}{2}\sqrt[3]{2\frac{4}{5}}.$
14. $\frac{a}{2}\sqrt{bc}.$
15. $ab\sqrt[3]{\frac{1}{abc}}.$
16. $(a+b)\sqrt{ab}.$
17. $2(a+b)\sqrt{\frac{a}{a+b}}.$
18. $\frac{1}{bc}\sqrt{a-b}.$
19. $\frac{a+b}{a-b}\sqrt{\frac{a-b}{a+b}}.$
20. $\frac{x+2}{x-2}\sqrt{1-\frac{4}{x+2}}.$

2. To change the order of a surd.

EXAMPLES

1. $\sqrt{3} = \sqrt[4]{3^2} = \sqrt[4]{9}.$

2. $3\sqrt[3]{2a^2} = 3\sqrt[9]{2^3a^6} = 3\sqrt[9]{8a^6}.$

To multiply or divide the order of a surd by a positive integer multiply or divide the exponent of each factor under the radical sign by the same number.

This process is made easier by using fractional exponents Thus, to change $\sqrt[3]{2a^2}$ to the ninth order,

$$\sqrt[3]{2a^2} = (2a^2)^{\frac{1}{3}} = (2a^2)^{3\cdot\frac{1}{9}} = (8a^6)^{\frac{1}{9}} = \sqrt[9]{8a^6}.$$

WRITTEN EXERCISES

1. Reduce the radicals to the fourth order:

$$\sqrt{2},\quad \sqrt{a},\quad 2\sqrt{ab},\quad \sqrt{\tfrac{1}{2}},\quad \frac{1}{2}\sqrt{\frac{b}{2a}},\quad a\sqrt{a+b}.$$

2. Reduce the above radicals to the sixth order.
3. Reduce the radicals to the ninth order:

$$\sqrt[3]{3},\quad \sqrt[3]{a},\quad 3\sqrt[3]{ab},\quad \sqrt[3]{\tfrac{1}{3}},\quad a\sqrt[3]{\frac{b}{2c}},\quad ab\sqrt[3]{x-y}.$$

4. Reduce the radicals to the second order:

$$\sqrt[4]{4},\quad \tfrac{1}{2}\sqrt[10]{a^5},\quad \sqrt[6]{\frac{1}{(3x)^3}},\quad a+b\sqrt[4]{(a-b)^2},\quad xy\sqrt[8]{(a-2b)^4}.$$

3. To reduce radicals to the same order.

Apply the processes in case 2 above to reduce each radical to the given order.

WRITTEN EXERCISES

1. Reduce to radicals of the eighth order: $\sqrt{3}$, $\sqrt[4]{2}$.
2. Reduce to radicals of the sixth order: $\sqrt{5}$, $\sqrt[3]{4}$.

Reduce to entire surds of the order named:

3. $\sqrt{2}$, $\sqrt[4]{3a}$, 8th order.
4. $\sqrt[6]{5}$ $2\sqrt[3]{a}$, 12th order.
5. $3\sqrt[5]{a^3x^4}$, $\sqrt{mn}$, 10th order.
6. $\sqrt{\frac{1}{2}}$, $\sqrt[4]{\frac{1}{2}x^3y}$, 8th order.
7. $p\sqrt[4]{p^3r}$, $\sqrt{\frac{1}{2}ax}$, 12th order.
8. $6\sqrt{5}$, $.3\sqrt[3]{.5}$, 6th order.
9. Reduce $\sqrt[3]{\frac{1}{2}a^2b}$, $7\sqrt[5]{\frac{a-b}{2}}$, to the 15th order.
10. Reduce $\sqrt{2}$, $\sqrt[4]{3}$, $\sqrt[5]{4}$, $\sqrt[10]{5}$, to the 20th order.

11. Reduce to radicals of the sixth order:

$$\sqrt{2},\quad \sqrt[3]{ab},\quad \sqrt{a+b}.$$

12. Reduce to radicals of the tenth order:

$$\sqrt{2},\quad \sqrt{ax},\quad 5\sqrt{2xy},\quad \sqrt[5]{a+b},\quad (a-b)\sqrt[5]{x-2y}.$$

Reduce to radicals of the same order:

13. $(x+y)\sqrt{x-y},\ \sqrt[3]{x-y}.$ **16.** $\sqrt[2n]{a},\ \sqrt{ab},\ \sqrt[n]{a^3b^3}.$

14. $\sqrt{a},\ \sqrt[4]{a^2-b^2},\ \sqrt{a-b}.$ **17.** $\sqrt[4p]{x},\ \sqrt{xy},\ \sqrt[p]{x-y}.$

15. $\sqrt{m},\ \sqrt[5]{m-1},\ \sqrt[10]{m-2}.$ **18.** $\sqrt[n]{a-b},\ \sqrt[5n]{a-b+c}.$

Which is the greater

19. $\sqrt[3]{2}$ or $\sqrt[6]{5}$? **20.** $\sqrt{2}$ or $\sqrt[3]{3}$? **21.** $\sqrt{2\frac{1}{2}}$ or $\sqrt[3]{4}$?

4. To reduce a radical to its simplest form.

EXAMPLES

1. $\sqrt{8a^3}=\sqrt{4a^2\cdot 2a}=2a\sqrt{2a}.$ Sec. 349.
2. $\sqrt[3]{2a^6x^5}=\sqrt[3]{(a^2x)^3\cdot 2x^2}=a^2x\sqrt[3]{2x^2}.$
3. $\sqrt{\frac{1}{2}}=\sqrt{\frac{2}{4}}=\sqrt{\frac{1}{4}\cdot 2}=\frac{1}{2}\sqrt{2}.$
4. $\sqrt[8]{a^2x^4}=\sqrt[4\times 2]{(ax^2)^2}=\sqrt[4]{ax^2}.$

Note that different exponents like the 2 and 3 in $\sqrt[6]{a^2b^3}$ must not be canceled from the common index.

356. The reduction of radicals is facilitated by the use of fractional exponents.

Consider Example 2 above:

$$\sqrt[3]{2a^6x^5}=(2a^6x^5)^{\frac{1}{3}}=2^{\frac{1}{3}}(a^6)^{\frac{1}{3}}\cdot(x^5)^{\frac{1}{3}}=2^{\frac{1}{3}}a^2\cdot(x^3\cdot x^2)^{\frac{1}{3}}$$
$$=2^{\frac{1}{3}}a^2x\,(x^2)^{\frac{1}{3}}=a^2x(2x^2)^{\frac{1}{3}}=a^2x\sqrt[3]{2x^2}.$$

Similarly, for Example 3:

$$\sqrt{\tfrac{1}{2}}=(\tfrac{1}{2})^{\frac{1}{2}}=(\tfrac{1}{4}\cdot 2)^{\frac{1}{2}}=\tfrac{1}{2}(2)^{\frac{1}{2}}=\tfrac{1}{2}\sqrt{2}.$$

Also for example 4:

$$\sqrt[8]{a^2x^4}=(a^2x^4)^{\frac{1}{8}}=(a^2)^{\frac{1}{8}}\ \ (x^4)^{\frac{1}{8}}=(a)^{\frac{1}{4}}\cdot(x^2)^{\frac{1}{4}}=(ax^2)^{\frac{1}{4}}=\sqrt[4]{ax^2}.$$

357. A radical is understood to be in its simplest form when:

(1) *No factor can be taken from under the radical sign* (*Examples 1 and 2*, Sec. 355, 4);

(2) *The radicand is integral* (*Example 3*).

(3) *The radicand is not a power whose exponent has a common factor with the order of the radical* (*Example 4*).

WRITTEN EXERCISES

Reduce to simplest form:

1. $\sqrt{18}$.
2. $\sqrt{27}$.
3. $\sqrt{20}$.
4. $\sqrt{32}$.
5. $2\sqrt{8}$.
6. $3\sqrt{12}$.
7. $5\sqrt{50}$.
8. $\sqrt[3]{16}$.
9. $3\sqrt[3]{32}$.
10. $3\sqrt{75}$.
11. $\sqrt[4]{320}$.
12. $\sqrt[3]{3000}$.
13. $\sqrt{\frac{1}{3}}$.
14. $3\sqrt{\frac{1}{2}}$.
15. $3\sqrt{\frac{1}{27}}$.
16. $2\sqrt[3]{\frac{1}{4}}$.
17. $5\sqrt[6]{\frac{1}{16}}$.
18. $\sqrt{a^2x^2y}$.
19. $\sqrt{\frac{ax^2y}{b}}$.
20. $\sqrt{\frac{1}{a}}$.
21. $\sqrt{\frac{b^2}{2x^3}}$.
22. $a\sqrt[3]{\frac{b}{2a}}$.
23. $4\sqrt[4]{15\frac{1}{2}}$.
24. $2\sqrt{\frac{a^4b^3}{2x^3}}$.
25. $\sqrt[2x]{a^{3x}b^{2x}}$.
26. $\frac{a}{b^n}\sqrt{\frac{b^{2n}}{a^5}}$.
27. $\sqrt[4]{\frac{81}{1296}a^{20}b^8c^{12}}$.
28. $\sqrt{1-(\frac{1}{3})^2}$.
29. $\sqrt{1+(\frac{1}{2})^2}$.
30. $\sqrt{3-(\frac{3}{4})^2}$.
31. $\sqrt{1-(\frac{2}{3})^2}$.
32. $3\sqrt{\frac{x^3}{27}}$.
33. $\sqrt{a^2-\left(\frac{a}{2}\right)^2}$.
34. $\sqrt{18+9\sqrt{2}}$.
35. $\sqrt{\frac{r^2-r^2\sqrt{3}}{2}}$.
36. $ab\sqrt{2+\frac{a^2+b^2}{ab}}$.
37. $\sqrt[3]{a^6-3a^3}$.
38. $\sqrt{7x^2-14xy+7y^2}$.
39. $(5x^2-10ax+5a^2)^{\frac{1}{2}}$.
40. $\sqrt{m^2-6}\,(m^4-6m^2)^{\frac{1}{2}}$.
41. $\sqrt[3]{a \div b} \cdot \sqrt[3]{a^2c \div b^2}$.
42. $(a-b)\sqrt{a^2-2ab+b^2}$.

43. $\sqrt{x^2+\left(\frac{x}{4}\right)^2}$.

44. $\sqrt[3]{p^3-\left(\frac{p}{2}\right)^3}$.

45. $m\sqrt[3]{\frac{10}{3m^2}}$.

46. $\sqrt[3]{27-81\sqrt{3}}$.

47. $\sqrt{(a^4-2a^3b+a^2b^2)\div b^2}$.

48. $ab\sqrt{3x^2+6xy+3y^2}$.

49. $(x+y)\sqrt{a^2b-a^3}$.

50. $\sqrt{p+\left(\frac{p}{c}\right)^2}$.

51. $\sqrt{\left(x+\frac{x}{2}\right)\left(x-\frac{x}{2}\right)}$.

358. Addition and Subtraction of Radical Expressions. Radicals can be united by addition or subtraction only when the same root is indicated and the expressions under the radical sign are the same in each.

359. When the expression cannot be put into this form the sum or the difference can only be indicated.

360. *To add or subtract radical expressions having the same radical part, add or subtract the coefficients of their radical parts.*

For example:

1. $2\sqrt{3}+3\sqrt{3}=5\sqrt{3}$. 2. $2\sqrt{12}+\sqrt{300}=4\sqrt{3}+10\sqrt{3}=14\sqrt{3}$.

3. Add $\sqrt{2}$, $-\sqrt{8}$, $\sqrt[3]{16}$, $\sqrt[3]{-54}$:

$$-\sqrt{8}=-2\sqrt{2};\ \sqrt[3]{16}=2\sqrt[3]{2};\ \sqrt[3]{-54}=-3\sqrt[3]{2}.$$

$$\therefore \sqrt{2}-\sqrt{8}+\sqrt[3]{16}+\sqrt[3]{-54}=\sqrt{2}-2\sqrt{2}+2\sqrt[3]{2}-3\sqrt[3]{2}=-\sqrt{2}-\sqrt[3]{2}.$$

Since only like radical parts may be added, it will be clear that $\sqrt{a}+\sqrt{b}$ does not mean $\sqrt{a+b}$. Show this by letting $a=9$, $b=16$.

WRITTEN EXERCISES

Find the sum:

1. $\sqrt{2}$, $\sqrt{8}$, $\sqrt{18}$.
2. $\sqrt{75}$, $-\sqrt{12}$, $-\sqrt{3}$.
3. $\sqrt{8}$, $\sqrt{5}$, $-\sqrt{18}$.
4. $\sqrt{128}$, $-2\sqrt{50}$, $\sqrt{72}$.
5. $\sqrt[3]{40}$, $-\sqrt[3]{320}$, $\sqrt[3]{135}$.
6. $8\sqrt{48}$, $-\frac{1}{2}\sqrt{12}$, $4\sqrt{27}$.
7. $\sqrt[3]{72}$, $-3\sqrt[3]{9}$, $6\sqrt[3]{243}$.
8. $\sqrt{6}$, $\sqrt{24}$, $\sqrt{63}$.
9. $\sqrt{108}$, $-\sqrt{12}$, $\sqrt{48}$.
10. $\sqrt{75}$, $\sqrt{48}$, $-\sqrt{27}$.
11. $\sqrt{80}$, $\sqrt{20}$, $-\sqrt{45}$.
12. $\sqrt{44}$, $-\sqrt{99}$, $\sqrt{121}$.
13. $5\sqrt{24}$, $-\sqrt{54}$, $3\sqrt{96}$.
14 $\sqrt[3]{27x^4}$, $-\sqrt[3]{64x^4}$, $\sqrt[3]{16x^4}$.

361. Multiplication of Radical Expressions containing Square Roots. In multiplying expressions containing indicated square roots, make use of the relation $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$.

EXAMPLES

1.
$$\begin{array}{llll} 3-4\sqrt{5} & & & \\ 6+2\sqrt{3} & & & \\ \hline 18-24\sqrt{5} & & & \\ & & 6\sqrt{3} & -8\sqrt{15} \\ \hline 18-24\sqrt{5} & + & 6\sqrt{3} & -8\sqrt{15}. \end{array}$$

2.
$$\begin{array}{ll} 2-\sqrt{3} & \\ 5+2\sqrt{3} & \\ \hline 10-5\sqrt{3} & \\ \quad\;\; 4\sqrt{3} & -6 \\ \hline 10-\;\;\sqrt{3} & -6=4-\sqrt{3}. \end{array}$$

WRITTEN EXERCISES

Multiply:

1. $2+\sqrt{5}$ by $2-\sqrt{5}$.
2. $1+\sqrt{3}$ by $2+\sqrt{5}$.
3. $2+\sqrt{3}$ by $2+\sqrt{3}$.
4. $\sqrt{2}+\sqrt{3}$ by $1-\sqrt{3}$.
5. $\sqrt{3}-\sqrt{5}$ by $\sqrt{3}+\sqrt{5}$.
6. $\sqrt{5}-\sqrt{6}$ by $\sqrt{5}-\sqrt{6}$.
7. $4+\sqrt{5}$ by $\sqrt{10}$.
8. $3-\sqrt{15}$ by $2+\sqrt{5}$.
9. $1+\sqrt{2}$ by $1-\sqrt{8}$.
10. $2\sqrt{3}-3\sqrt{5}$ by $\sqrt{3}-\sqrt{5}$.
11. $\sqrt{14}+\sqrt{7}$ by $\sqrt{8}-\sqrt{21}$.
12. $\sqrt{5}-\sqrt{48}$ by $\sqrt{5}+\sqrt{12}$.

362. Division of Square Roots. The quotient of the square roots of two numbers is the square root of the quotient of the numbers. In symbols, $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$.

Thus, $\dfrac{\sqrt{5}}{\sqrt{6}}=\sqrt{\dfrac{5}{6}}$, because, multiplying each member by itself,

$$\frac{\sqrt{5}}{\sqrt{6}}\cdot\frac{\sqrt{5}}{\sqrt{6}}=\sqrt{\frac{5}{6}}\cdot\sqrt{\frac{5}{6}},$$

or
$$\frac{\sqrt{5}\cdot\sqrt{5}}{\sqrt{6}\cdot\sqrt{6}}=\sqrt{\frac{5}{6}}\cdot\sqrt{\frac{5}{6}},$$

or
$$\frac{\sqrt{5^2}}{\sqrt{6^2}}=\sqrt{\left(\frac{5}{6}\right)^2},$$

or
$$\frac{5}{6}=\frac{5}{6}, \text{ which is evidently true.}$$

ORAL EXERCISES

Read each of the following as a fraction under one radical sign:

1. $\dfrac{\sqrt{3}}{\sqrt{5}}$.
2. $\dfrac{\sqrt{3}}{\sqrt{7}}$.
3. $\dfrac{\sqrt{5}}{\sqrt{7}}$.
4. $\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{1}}{\sqrt{2}}$.
5. $\dfrac{1}{\sqrt{5}}$.
6. $\dfrac{\sqrt{2}}{\sqrt{8}}$.
7. $\dfrac{\sqrt{5}}{\sqrt{15}}$.
8. $\dfrac{\sqrt{10}}{\sqrt{20}}$.

363. In the four processes with radicals explained in Secs. 359–362 we have admitted only the exact values of the radical expressions, but if we accept approximate values of the radicals, we may further simplify the results obtained, and this is often done for practical purposes.

EXAMPLE

Each of the two legs of a right triangle is 1 in., and the hypotenuse is $\sqrt{2}$ in. Find the perimeter.

The exact result is $(1 + 1 + \sqrt{2})$ in. $= (2 + \sqrt{2})$ in.

If we accept an approximate result, taking three decimal places as the degree of accuracy, we have

$(2 + 1.414)$ in. $= 3.414$ in. Secs. 333, 334.

WRITTEN EXERCISES

Find to two decimal places the value of:

1. $2+\sqrt{3}$.
2. $3-\sqrt{3}$.
3. $3-\sqrt{2}$.
4. $4-\sqrt{2}$.
5. $\sqrt{2}+\sqrt{3}$.
6. $\sqrt{3}-\sqrt{2}$.
7. $\frac{1}{2}\sqrt{2}-1$.
8. $\frac{1}{2}\sqrt{3}+2$.
9. $\sqrt{5}-2$.

Find to three decimal places the value of:

10. $\frac{1}{3}\sqrt{3}$.
11. $\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{2}$.
12. $\dfrac{3-\sqrt{2}}{2}$.
13. $\sqrt{2}-\sqrt{3}$.
14. $\sqrt{5}-\sqrt{2}$.
15. $\sqrt{6}-\sqrt{3}$

16. Find x to two decimal places in the equation $x^2 - 3 = 0$.

17. The altitude of an equilateral triangle with sides 1 in. is $\frac{1}{2}\sqrt{3}$ in. Find the altitude to two decimal places.

18. Find the perimeter of a right triangle to two decimal places whose sides are 1 in., 2 in., and $\sqrt{5}$ in.

19. Find the perimeter of a right triangle to two decimal places whose sides are 1 ft., $\frac{1}{2}$ ft., and $\frac{1}{2}\sqrt{3}$ ft.

364. Rationalizing the Denominator. Multiplying both numerator and denominator of a fraction by an expression that will make the denominator rational is called **rationalizing the denominator.**

Thus, multiplying both numerator and denominator of $\frac{\sqrt{3}}{\sqrt{2}}$ by $\sqrt{2}$, we obtain

$$\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{2}\cdot\sqrt{3}}{\sqrt{2}\cdot\sqrt{2}} = \frac{\sqrt{6}}{2}.$$

WRITTEN EXERCISES

Rationalize the denominator of:

1. $\frac{1}{\sqrt{2}}$.
2. $\frac{1}{\sqrt{3}}$.
3. $\frac{1}{\sqrt{5}}$.
4. $\frac{2}{\sqrt{3}}$.
5. $\frac{3}{\sqrt{7}}$.
6. $\frac{\sqrt{2}}{\sqrt{3}}$.
7. $\frac{\sqrt{5}}{\sqrt{7}}$.
8. $\frac{5}{\sqrt{3}}$.
9. $\frac{\sqrt{5}}{\sqrt{3}}$.
10. $\frac{10}{\sqrt{5}}$.
11. $\frac{6}{\sqrt{3}}$.
12. $\frac{8}{\sqrt{2}}$.

365. Rationalizing Factors. When the denominator is of the form $\sqrt{a}+\sqrt{b}$ or $a+\sqrt{b}$, the rationalizing factor is the same binomial with the connecting sign changed, often called the **conjugate binomial.**

It is not necessary in elementary algebra to take up the rationalizing of more complicated denominators.

EXAMPLES

1. Rationalize the denominator in $\frac{3}{2-\sqrt{5}}$.

The conjugate of $2-\sqrt{5}$ is $2+\sqrt{5}$.

Then, $\frac{3}{2-\sqrt{5}} = \frac{3(2+\sqrt{5})}{(2-\sqrt{5})(2+\sqrt{5})} = \frac{6+3\sqrt{5}}{4-5} = -(6+3\sqrt{5})$.

2. Rationalize the denominator in $\dfrac{2+\sqrt{3}}{\sqrt{3}+\sqrt{5}}$.

The conjugate of $\sqrt{3}+\sqrt{5}$ is $\sqrt{3}-\sqrt{5}$.

Then, $\dfrac{2+\sqrt{3}}{\sqrt{3}+\sqrt{5}}=\dfrac{(2+\sqrt{3})(\sqrt{3}-\sqrt{5})}{(\sqrt{3}+\sqrt{5})(\sqrt{3}-\sqrt{5})}=\dfrac{2\sqrt{3}+3-2\sqrt{5}-\sqrt{15}}{3-5}$

$$=-\frac{3+2\sqrt{3}-2\sqrt{5}-\sqrt{15}}{2}.$$

3. Rationalize the denominator of $\dfrac{3-\sqrt{5}}{\sqrt{2}-\sqrt{5}+\sqrt{3}}$.

First multiply the terms by the expression $\sqrt{2}+(\sqrt{5}-\sqrt{3})$.

Then, we have $\dfrac{(3-\sqrt{5})(\sqrt{2}+\sqrt{5}-\sqrt{3})}{2-(\sqrt{5}-\sqrt{3})^2}=\dfrac{(3-\sqrt{5})(\sqrt{2}+\sqrt{5}-\sqrt{3})}{-6+2\sqrt{15}}$.

Then, multiply by the conjugate $-6-2\sqrt{15}$ and simplify as usual.

WRITTEN EXERCISES

Rationalize the denominators:

1. $\dfrac{2+\sqrt{3}}{3+\sqrt{3}}$.

2. $\dfrac{3\sqrt{3}+2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

3. $\dfrac{1}{1-\sqrt{2}}$.

4. $\dfrac{3}{2-\sqrt{5}}$.

5. $\dfrac{5}{\sqrt{3}+\sqrt{7}}$.

6. $\dfrac{2-\sqrt{5}}{3-\sqrt{5}}$.

7. $\dfrac{3}{\sqrt{5}+\sqrt{2}}$.

8. $\dfrac{\sqrt{5}+2\sqrt{2}}{4-2\sqrt{2}}$.

9. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.

10. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$.

11. $\dfrac{2+4\sqrt{7}}{2\sqrt{7}-1}$.

12. $\dfrac{2\sqrt{15}-6}{\sqrt{5}+2\sqrt{2}}$.

13. $\dfrac{1}{1-\sqrt{2}+\sqrt{3}}$.

14. $\dfrac{1+\sqrt{2}}{\sqrt{3}-\sqrt{5}+4}$.

15. $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}+\sqrt{5}}$.

16. $\dfrac{3-\sqrt{5}+\sqrt{7}}{3+\sqrt{5}-\sqrt{7}}$.

17. $\dfrac{3}{4-(\sqrt{6}+\sqrt{5})}$.

18. $\dfrac{c}{2-\sqrt{a-b}}$.

19. $\dfrac{1}{a-\sqrt{2b+c}}$.

20. $\dfrac{a+\sqrt{b}}{2\sqrt{a}-\sqrt{b}+\sqrt{c}}$.

21. $\dfrac{\sqrt{a^2-1}+\sqrt{a^2+1}}{\sqrt{a^2+1}-\sqrt{a^2-1}}$.

22. $\dfrac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$.

23. $\dfrac{\sqrt{x^2+x+1}-1}{\sqrt{x^2+x+1}+1}$.

24. $\dfrac{\sqrt{p+q}-\sqrt{p-q}}{\sqrt{p+q}+\sqrt{p-q}}$.

366. Chapter XII dealt only with rational factors, but the properties of radicals make it possible to find irrational factors.

EXAMPLES

1. Factor: $2x^2-1$.

This is the difference of two squares, if we admit radicals, because 2 is the square of $\sqrt{2}$.

Then, $2x^2-1=(\sqrt{2}x-1)(\sqrt{2}x+1)$.

2. Factor: x^2-5.

Using radicals, $x^2-5=x^2-(\sqrt{5})^2$.

Then, $x^2-5=(x-\sqrt{5})(x+\sqrt{5})$.

3. Similarly, $x-y=(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$.

4. $x-y=(\sqrt[3]{x}-\sqrt{y})(\sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2})$.

367. Irrational factors may be used to solve equations.

Thus, In $2x^2-1=0$, $(\sqrt{2}x-1)(\sqrt{2}x+1)=0$.

Therefore, $\sqrt{2}x-1=0$, and $\sqrt{2}x+1=0$.

Solving, $x=\dfrac{1}{\sqrt{2}}=\dfrac{1}{2}\sqrt{2}$, also $x=-\dfrac{1}{\sqrt{2}}=-\dfrac{1}{2}\sqrt{2}$.

WRITTEN EXERCISES

Factor:

1. x^2-3.
2. $4x^2-5$.
3. $8x^2-1$.
4. $5x^2-4$.
5. $6x^2-9$.
6. $2a^2-b$.
7. z^2-7.
8. $a-b^2$.

9. What factor taken with $\sqrt[3]{a}-\sqrt[3]{b}$ makes $a-b$?

10. What factor taken with $x-\sqrt[3]{2}$ makes x^3-2?

Solve.

11. $y^2-2=0$.
12. $z^2-3=0$.
13. $2p^2-1=0$.
14. $ax^2-b=0$.
15. $2m^2-5=0$.
16. $2x^2-12=0$.

REVIEW

WRITTEN EXERCISES

Simplify:

1. $\dfrac{\sqrt{5}}{\sqrt{60}}$.
2. $\sqrt{\frac{27}{8}} - \sqrt{\frac{3}{8}}$.
3. $\sqrt{6} \cdot \sqrt{125}$.
4. $\dfrac{\sqrt{10}}{\sqrt{40}}$.
5. $\sqrt{50} + \sqrt{128}$.
6. $\sqrt{3} \div \sqrt{5}$.
7. $\dfrac{3\sqrt{3} + 2\sqrt{2}}{\sqrt{3} - \sqrt{2}}$.
8. $\sqrt{6} \div \sqrt{2}$.
9. $(2 + \sqrt{3})^2$.
10. $(5 + \sqrt{7})(5 - \sqrt{7})$.
11. $(2\sqrt{3} + 3\sqrt{5}) \div \sqrt{15}$.
12. $(\sqrt{6} + \sqrt{15})(\sqrt{8} - \sqrt{20})$.

Express with rational denominators, and with at most one radical sign in the dividend:

13. $\sqrt{12} \div \sqrt{3}$.
14. $\sqrt{7} \div \sqrt{11}$.
15. $2\sqrt{24} \div 2\sqrt{6}$.
16. $2 \div 3\sqrt{5}$.
17. $\dfrac{4}{\sqrt{5} - 1}$.
18. $1 \div (\sqrt{2} - 10)$.
19. $\sqrt{2} \div (\sqrt{2} - \sqrt{3})$.
20. $(2\sqrt{6} + 5\sqrt{12}) \div \sqrt{6}$.
21. $(5\sqrt{18} - 8\sqrt{50}) \div 2\sqrt{2}$.
22. $\dfrac{8 - 5\sqrt{2}}{3 - 2\sqrt{2}}$.
23. $\dfrac{3 + \sqrt{5}}{3 - \sqrt{5}}$.
24. $\dfrac{\sqrt{3}}{\sqrt{5} - \sqrt{3}}$.
25. $\dfrac{2 + 4\sqrt{7}}{2\sqrt{7} - 1}$.
26. $\dfrac{4\sqrt{7} + 3\sqrt{2}}{5\sqrt{2} + 2\sqrt{7}}$.

Solve by factoring:

27. $3x^2 - 1 = 0$.
28. $4x^2 - 2 = 0$.
29. $5x^2 - a = 0$.
30. $3x^3 - 9x = 0$.

31. Computing the square root to two decimal places, find r in the equation $\pi r^2 = 8$. (Use $\pi = 3.1416$.)

32. Find t to three decimal places, using $g = 32$, in the equation $48 = \frac{1}{2}gt^2$.

33. Find the value of $\frac{1}{3}h(B + b + \sqrt{Bb})$, taking $h = 5$, $b = 3$, $B = 8$, and computing the radical to two decimal places.

34. Find the value of the expression in Exercise 33 when $h = 9$, $b = 8$, $B = 17$, computing the radical to 3 decimal places.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define *rational numbers;* also *irrational numbers.* Secs. 337, 338.

2. Define a *radical;* also a *surd;* also a *radical expression.* Secs. 339, 341, 346.

3. What sign is chosen for radicals of even order? Sec. 340.

4. What is the *radicand?* Illustrate. Sec. 341.

5. What is an *entire surd?* A *mixed surd?* Sec. 343.

6. What is meant by the *order* of a surd or radical? State a *quadratic* surd; a *cubic* surd; a *biquadratic* surd. Sec. 344, 345.

7. What is another expression for the product of two square roots? Sec. 348.

8. How may factors be placed under the radical sign? Sec. 350.

9. How is a mixed radical reduced to an entire radical? Sec. 355.

10. Illustrate how to change the order of a radical. Sec. 355.

11. State how to *simplify* a radical. Sec. 355.

12. Illustrate how the use of fractional exponents simplifies the process of reduction of radicals. Sec. 356.

13. What radicals may be added? How are they added or subtracted? Secs. 358–360.

14. What is another expression for the quotient of two square roots? Sec. 362.

15. What is meant by *rationalizing the denominator* of a fraction containing radicals? Sec. 364.

CHAPTER XXII

QUADRATIC EQUATIONS

RATIONAL

368. The general form of a quadratic equation is

(1) $$ax^2 + bx + c = 0.$$

By dividing this equation by a, the coefficient of x^2, the equation becomes $x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$

Putting $p = \frac{b}{a}$, $q = \frac{c}{a}$ to replace the fractional forms, the equation becomes

(2) $$x^2 + px + q = 0$$

which is also a general form for quadratic equations.

369. Kinds of Quadratic Equations. The equation $ax^2 + bx + c = 0$ is said to be a **complete quadratic equation** when neither b nor c is zero; that is, when there are three terms, one containing x^2, another x, and a term without x (absolute term). In any other case it is called an **incomplete quadratic equation.**

370. A quadratic in which the first power of the unknown does not occur is called a **pure** quadratic, and one in which it occurs is called an **affected** quadratic.

Thus, $x^2 + 5x - 2 = 0$ is a complete quadratic equation, while $2x^2 - 5 = 0$, and $x^2 + 7x = 0$ are incomplete quadratics. The second equation is a pure quadratic and the others are affected quadratics.

371. Solution by Completing the Square. The following example shows a general method for solving quadratic equations with one unknown:

EXAMPLE

Solve: $x^2+8x+7=0.$ *(1)*

Transposing the absolute term, $x^2+8x=-7.$ *(2)*

Making the left member a square by adding 16 to *both* members, $x^2+8x+16=9.$ *(3)*

$\therefore (x+4)^2=9.$ *(4)*

Extracting the square root of both members, $x+4=\pm 3.$ *(5)*

$\therefore x=-1 \text{ or } -7.$ *(6)*

The process consists of two essential parts:

(1) *Making the left member a square while the right member does not contain the unknown.*

This is called completing the square.

It is based upon the relation $(x+a)^2=x^2+2ax+a^2$ (Sec. 141, p. 103) in which it appears that the last term, a^2, is the square of one half of the coefficient of x.

(2) *Extracting the square roots of both members and solving the resulting linear equations.*

WRITTEN EXERCISES

Solve completely:

1. $x^2+8x=9.$
2. $x^2+4x=12.$
3. $x^2+12x=-11.$
4. $x^2-8x=9.$
5. $x^2+10x=11.$
6. $x^2+5x=\frac{11}{4}.$
7. $x^2-20x=-75.$
8. $x^2-6x=7.$
9. $z^2-40z=41.$
10. $w^2-w=\frac{3}{4}.$
11. $t^2-2t=8.$
12. $u^2-u=-\frac{1}{4}.$
13. $v^2+3v=\frac{-5}{4}.$
14. $s^2-18s=19.$

312. Approximate Roots Expressed Decimally. Square roots which cannot be found exactly may be indicated and should be so used for checking, but the roots of quadratic equations expressing the solution of practical problems are often irrational and require approximating. The nature of the problem must determine to how many decimal places the result should be expressed, but for practice we shall use two decimal places.

EXAMPLE

Solve: $x^2 - 6x + 3 = 0.$ *(1)*

Rearranging and completing the square, $x^2 - 6x + 9 = 6.$ *(2)*

$\therefore x - 3 = \pm\sqrt{6}.$ *(3)*

$\therefore x = 3 + \sqrt{6}$ and $3 - \sqrt{6}.$ *(4)*

To test this work we must substitute $3 + \sqrt{6}$ and $3 - \sqrt{6}$ in equation (1). But, by taking $\sqrt{6}$ to the nearest hundredth, or as 2.45 these roots are $3 + \sqrt{6} = 5.45$ and $3 - \sqrt{6} = 0.55$. These roots will not check since they are only approximations, but they are definite rational numbers and sufficiently accurate for many practical purposes.

WRITTEN EXERCISES

Solve, computing all irrational roots to two decimal places.

1. $y^2 - y = 1.$
2. $x^2 + x = 5.$
3. $x^2 - 6x = -1.$
4. $x^2 - 6x = -3.$
5. $x^2 + 5x = 6.$
6. $x^2 - 4x = 20.$
7. $x^2 + \frac{1}{2}x = \frac{3}{2}.$
8. $x^2 - \frac{1}{4}x = \frac{1}{2}.$
9. $z^2 - 16z = -15.$
10. $t^2 - 2t = 6.$
11. $u^2 - u = 1.$
12. $x^2 + 13x = -30.$
13. $x^2 + 6x = -4.$
14. $x^2 - 8x = -8.$
15. $x^2 + 13x = \frac{3}{4}.$
16. $x^2 + 15x = -25.$
17. $x^2 - 8x = 3.$
18. $m^2 + 8m = 4.$
19. $x^2 + 10x = 1.$
20. $x^2 - 5x + 6 = 0.$
21. $x^2 - 8x = 7.$
22. $x^2 + 10x = 5.$
23. $p^2 - 6p = -1.$
24. $x^2 - 6x = 13.$
25. $x^2 + 5x = 7.$
26. $x^2 - 10x = 25.$
27. $x^2 - 9x = \frac{3}{2}.$
28. $x^2 - \frac{3}{4}x = \frac{1}{2}.$
29. $z^2 - 16z = 9.$
30. $t^2 - 8t = 6.$
31. $u^2 - u = 5.$
32. $x^2 + 3x = 10.$
33. $m^2 - 4m = 1.$
34. $w^2 + 5w + 6 = 0.$
35. $t^2 + 9t + 20 = 0.$
36. $v^2 - v - 20 = 0.$
37. $x^2 - x - 42 = 0.$
38. $x^2 - 5x - 84 = 0.$
39. $u^2 + 19u + 84 = 0.$
40. $z^2 - 9z + 14 = 0.$

373. Solution by Formula. When the coefficient of x^2 is not unity, the equation must be divided by that coefficient before using the method above to complete the square.

If the square of the general equation $x^2+px+q=0$ be completed, we have

$$x^2+px+\frac{p^2}{4}=\frac{p^2}{4}-q. \qquad \text{Sec. 371.}$$

Extracting the square root of both members,

$$x+\frac{p}{2}=\pm\sqrt{\frac{p^2}{4}-q}, \qquad \text{Sec. 151.}$$

or,

$$x=-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q}. \qquad \text{Sec. 364.}$$

If the general equation $a^2+bx+c=0$ be divided by a and solved as above the roots are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

These results are general formulas for finding the values of x in any quadratic equation.

EXAMPLE

Solve: $$5x^2+7x-2=0.$$

Dividing by 5, $$x^2+\tfrac{7}{5}x-\tfrac{2}{5}=0.$$

Here $p=+\tfrac{7}{5}$ and $q=-\tfrac{2}{5}$.

Hence, $\frac{p}{2}=\frac{7}{10}$ and $\sqrt{p^2-4q}=\sqrt{\frac{49}{25}-4\left(-\frac{2}{5}\right)}$,

and
$$x=-\tfrac{7}{10}\pm\tfrac{1}{2}\sqrt{\tfrac{49}{25}+\tfrac{8}{5}}$$
$$=-\tfrac{7}{10}\pm\tfrac{1}{10}\sqrt{89}.$$

WRITTEN EXERCISES

Solve:

1. $4x^2+6x-4=0.$
2. $9x^2+15x+6=0$
3. $4x^2-2x-2=0.$
4. $9x^2+3x-6=0.$
5. $25z^2-20z+4=0.$
6. $3x^2-7x-20=0.$
7. $4x^2+12x-55=0.$
8. $9w^2+6w-35=0.$
9. $9v^2-39v+22=0$
10. $4y^2-12y=91.$
11. $16t^2-8t=15.$
12. $4z^2+20z=-21.$

13. $6x^2 + x = 12.$

14. $6x^2 = -5x + 4.$

15. $12x^2 = 5x + 2.$

16. $9x^2 = 18x - 5.$

Clear each of the following of fractions and solve the resulting equation:

17. $\dfrac{120}{x+3} = \dfrac{120}{x} - 2.$

18. $\dfrac{5x}{x+4} - \dfrac{3x-2}{2x-3} = 2.$

19. $x - \dfrac{x^3-8}{x^2+5} = 2.$

20. $\dfrac{1}{3} + \dfrac{1}{3+x} + \dfrac{1}{3+2x} = 0.$

21. $\dfrac{x+22}{3} - \dfrac{4}{x} = \dfrac{9x-6}{2}.$

22. $\dfrac{x+2}{x-1} - \dfrac{4-x}{2x} = 2\tfrac{1}{8}.$

23. $\dfrac{48}{x+3} = \dfrac{165}{x+10} - 5.$

24. $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{13}{6}.$

25. $\dfrac{2x}{x-4} + \dfrac{2x-5}{x-3} = \dfrac{25}{3}.$

26. $\dfrac{3x-7}{x} + \dfrac{4x-10}{x+5} = \dfrac{7}{2}.$

27. $\dfrac{4x+7}{19} + \dfrac{5-x}{3+x} = \dfrac{4x}{9}.$

28. $\dfrac{3}{5-x} + \dfrac{1}{4-x} = \dfrac{6}{x+2}.$

29. The perimeter of a rectangular field is 100 yd. and its area is 600 sq. yd. Find its length and breadth.

Solution. 1. Let x be the length of the field.

2. Then $\dfrac{600}{x}$ is its width, and

3. $2x + \dfrac{2 \cdot 600}{x}$ is its perimeter, being twice the sum of its sides.

4. $\therefore 2x + \dfrac{2 \cdot 600}{x} = 100$, by the conditions of the problem.

5. $\therefore x^2 - 50x + 600 = 0$, simplifying (4).

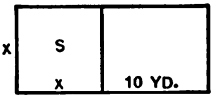
6. $\therefore (x-20)(x-30) = 0$, factoring (5).

7. $\therefore x = 20,\ x = 30$, solving (6).

8. If 20 yd. be taken as the length, the width is 30 yd., by step (2).

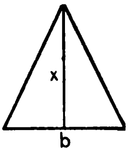
9. If 30 yd. be taken as the length, the width is 20 yd., by step (2).

30. The area of the whole plot shown in the diagram is 96 sq. yd. What is the length of a side of the square (s)?



31. The sum of two unequal sides of a rectangular court is 19 yd.; the sum of the areas of the squares

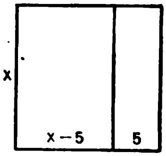
on these two sides is 181 sq. yd. What are the dimensions of the court?



32. A triangle whose area is 200 sq. yd. has its altitude equal to its base. Find the base of the triangle, using the formula, area = $\frac{1}{2}$ base × altitude.

33. The breadth of a room is 4 ft. more than its height and 20 ft. less than its length. The cost of cleaning and painting the side walls was $20.70, at $2\frac{1}{2}$¢ per square foot. Find the dimensions of the room.

34. A partition is built 5 yd. from one side of a square room as shown in the diagram; the area of the floor remaining is 24 sq. yd. What are the dimensions of the floor?



35. One side of a rectangle is $\frac{3}{4}$ as long as the other; if 10 ft. be added to the shorter side and 10 ft. be subtracted from the longer side, the area will not be changed. Find the dimensions of the original rectangle.

36. If there are s subscribers in a telephone exchange, the total number of different connections of any subscriber with any other is $\frac{s(s-1)}{2}$. If in an exchange the total number of different connections is 3240, find the number of subscribers.

37. A builder used steel bars weighing 120 lb. each for a certain purpose. By changing the mode of support, he found that he could get the same service from bars weighing 2 lb. less per running foot, but 2 ft. longer than the original bars. The new bars also weighed 120 lb. Find the length of the original bars.

374. Quadratic Forms. Certain equations of higher degree have the form of the quadratic equation and may be solved like a quadratic. These are said to be in **quadratic form.**

Only a few of these will be given here because their roots are usually complex in form and equations of this kind are given in Chapter XXX.

EXAMPLES

1. Solve: $x^4 - 13x^2 + 36 = 0.$ *(1)*

Let $y = x^2$, then $y^2 - 13y + 36 = 0.$ *(2)*

Factoring, $(y - 9)(y - 4) = 0.$ *(3)*

Hence, $y = 9$ and $y = 4.$ *(4)*

Since $y = x^2$ $x^2 = 9$ and $x^2 = 4.$ *(5)*

$\therefore x = +3, -3$ and $x = +2, -2.$ *(6)*

All of these roots check, making four values of x for the given equation of the fourth degree.

2. Solve: $x^4 - 9x^2 + 8 = 0.$ *(1)*

Let $y = x^2$; then the given equation becomes, $y^2 - 9y + 8 = 0.$ *(2)*

Solving for y, $y = 8$ or $1.$ *(3)*

Therefore $x^2 = 8.$ *(4)*

Or, $x^2 = 1.$ *(5)*

Solving *(4)*, *(5)*, $x = \pm\sqrt{8}, \pm 1.$ *(6)*

Test these four values of x by substitution in the given equation.

WRITTEN EXERCISES

Solve and test:

1. $x^4 - 5x^2 + 4 = 0.$
2. $x^4 - 17x^2 + 16 = 0.$
3. $36x^4 - 13x^2 + 1 = 0.$
4. $4y^4 - 17y^2 + 4 = 0.$
5. $R^4 - 18R^2 + 81 = 0.$
6. $(x-1)^4 - 5(x-1)^2 + 4 = 0.$
7. $(y-5)^4 - 17(y-5)^2 + 16 = 0.$
8. $(t-3)^4 - 11(t-3)^2 - 42 = 0.$

RADICAL EQUATIONS

375. *To solve equations in which only a single square root occurs, transpose so that the square root constitutes one member. Square both members and solve the resulting equation.*

EXAMPLE

Solve: $2x - 3 = \sqrt{x^2 + 6x - 6}.$ *(1)*

Squaring both members, $4x^2 - 12x + 9 = x^2 + 6x - 6.$ *(2)*

Collecting terms, $3x^2 - 18x + 15 = 0.$ *(3)*

Solving *(3)*, $x = 5$ or $1.$ *(4)*

TEST. By trial, it appears that 5 satisfies the given equation, taking the radical as positive, while 1 satisfies the equation $2x - 3 = -\sqrt{x^2 + 6x - 6}.$

1. It must be remembered that the equation resulting from squaring will usually not be *equivalent* to the given equation (Sec. 172, p. 128). It may have additional roots, and substitution must determine which of the values found satisfy the given equation.

2. In order that the given problem may be definite, the radical must be taken with a given sign. If every possible square root is meant, two different equations are really given for solution. Thus, unless restricted, $2x = \sqrt{4 - 6x}$ is a compact way of uniting the two different equations, $2x = +\sqrt{4 - 6x}$, and $2x = -\sqrt{4 - 6x}$. If solved as indicated above, it appears that the first is satisfied when $x = \frac{1}{2}$, the second when $x = -2$.

3. In the exercises of the following set the radical sign is to be understood to mean the *positive* square root.

376. Extraneous Roots. The ambiguity of signs is removed by squaring; hence, the resulting equation appears to have more roots than the original. These roots of the final equation which do not satisfy the original are called **extraneous roots**, and should be omitted.

WRITTEN EXERCISES

Solve, and test only rational roots:

1. $x = \sqrt{10x + 7}$.
2. $x = \sqrt{b + x - bx}$.
3. $3x - 7\sqrt{x} = -2$.
4. $\sqrt{x + 4} - x = 4$.
5. $\dfrac{\sqrt{x} - 1}{3} = \dfrac{x}{16}$.
6. $x + 5\sqrt{37 - x} = 43$.
7. $\sqrt{x + 5} = x - 7$.
8. $\sqrt{2x + 7} = \dfrac{x}{3}$.
9. $x + \sqrt{x + 5} = 2x - 1$.
10. $30 = x - 29\sqrt{x}$.
11. $x = 2 + \sqrt{3 - 11x}$.
12. $x - \sqrt{x - 2} = 2$.
13. $x + 1 = \sqrt{x + 3}$.
14. $\sqrt{100 - x^2} = 10 - x$.
15. $x + \sqrt{2x - x^2} = 2$.
16. $\dfrac{x + 2}{2} - \dfrac{x + 1}{3} - \dfrac{\sqrt{2x + 1}}{2} = 0$.

377. If the equation involves but one radical, the method of Sec. 375 can be used; but when more than one radical occurs, successive squaring may be necessary.

EXAMPLE

Solve: $\sqrt{2x+6}+\sqrt{3x+1}=8.$ (*1*)

Rearranging, $\sqrt{2x+6}=8-\sqrt{3x+1}.$ (*2*)

Squaring, $2x+6=64-16\sqrt{3x+1}+3x+1.$ (*3*)

Collecting, $16\sqrt{3x+1}=x+59.$ (*4*)

Squaring again and collecting, $x^2-650x+3225=0.$ (*5*)

Solving, $x=5$, or 645. (*6*)

Test. Trial shows that the first of these values satisfies the given equation, and it is obvious by inspection that the second cannot satisfy the equation.

378. Sometimes it is best first to transform the given expression.

EXAMPLE

Solve: $2x+\sqrt{4x^2+9}=-\dfrac{2x+1+\sqrt{5x+6}}{2x-\sqrt{4x^2+9}}.$ (*1*)

Clearing of fractions, $9=2x+1+\sqrt{5x+6}.$ (*2*)

Rearranging, $8-2x=\sqrt{5x+6}.$ (*3*)

Squaring and collecting, $4x^2-37x+58=0.$ (*4*)

Hence, $x=2$, or $7\frac{1}{4}$. (*5*)

Test. By trial, 2 is seen to satisfy the given equation. To avoid the complete work of substituting $7\frac{1}{4}$, we note that every root of (1) must satisfy (3). If $x=7\frac{1}{4}$, the left member of (3) is negative and the right member positive. Hence, $7\frac{1}{4}$ is an extraneous root (Sec. 376) and should be disregarded. It would satisfy (3) if the radical had the negative sign.

WRITTEN EXERCISES

Solve:

1. $x-\sqrt{3x+10}=6.$
2. $\sqrt{x}-\sqrt{x-15}=1.$
3. $\sqrt{x^2-5}+\dfrac{6}{\sqrt{x^2-5}}=5.$
4. $x+2+(x+2)^{\frac{1}{2}}=20.$
5. $\sqrt{x+7}+\sqrt{3x-2}=\dfrac{4x+9}{\sqrt{3x-2}}.$

6. $4\sqrt{x^2+5x+6}=\sqrt{5}\cdot\sqrt{x^2+5x+6}$.

7. $\sqrt{x^2-a^2x^2}=\sqrt{x^4+b^4x^2}$.

8. $\sqrt{x}+\sqrt{a-x}=\sqrt{a+b}$.

9. $\sqrt{6z+6}-\sqrt{3z+1}=\sqrt{5z-21}$.

10. $\sqrt{4+z}+\dfrac{10}{\sqrt{5z}}=\sqrt{5z}$.

11. $\sqrt{\dfrac{2x^2-2x+1}{2x^2+2x+1}}-2=0$.

379. An equation whose radicals have been removed by squaring may be linear.

WRITTEN EXERCISES

Solve and test:

1. $\sqrt{y}-\sqrt{y-8}=\dfrac{2}{\sqrt{y-8}}$.

2. $\sqrt{x-1}-\sqrt{x}=\dfrac{2}{\sqrt{x}}$.

3. $\sqrt{x-1}-\sqrt{a-1}=0$.

4. $\dfrac{1}{1-x}+\dfrac{1}{1+\sqrt{x}}=\dfrac{1}{1-\sqrt{x}}$.

REVIEW

WRITTEN EXERCISES

Solve, approximating any irrational roots to two decimal places:

1. $x^2+7x=8$.
2. $3x^2=48$.
3. $x^2+25x=-100$.
4. $(3x+4)^2=96$.
5. $x^2-25x+144=0$.
6. $x^2+3x-28=0$.
7. $x^2-13x=68$.
8. $x^2-12x+27=0$.
9. $x^2+111x=3400$.
10. $5x^2+13x=370$.
11. $3z^2-2z=1$.
12. $y^2+4ay-2=0$.
13. $t^2+3t-6=0$.
14. $13x^2-39x=0$.
15. $a^2x^2-abx=0$.
16. $(a+b)x^2+cx=0$.
17. $m^2x^2-(m+r)x=0$.
18. $ax^2-bx=cx^2+dx$.
19. $\frac{1}{2}x^2=14-3x^2$.
20. $x^2+5=\frac{10}{7}x^2-16$.

21. $\dfrac{2x^2+4}{9}=x+1.$

22. $x-2+\sqrt{2-x}=0.$

23. $x^2+\frac{10}{3}x=19.$

24. $5-3x+\frac{1}{4}x^2=0.$

25. $x^2-7x=0.$

26. $y^2-ay=c.$

27. $x^2-12=30+x.$

28. $12x^2-3x=7x^2+2x.$

29. $10x^2+5x=15x^2+9x.$

30. $(a+b)^2x^2+(a+b)x=0.$

31. $\dfrac{6}{x}+4=2x.$

32. $x+\sqrt{9-x^2}=4.$

33. $x^4-10x^2+9=0.$

34. $(z-3)^4-5(z-3)=-4.$

35. $(w-7)^4-(w-7)^2=2.$

36. The perimeter of a rectangular field is 200 ft., and its area is 2400 sq. ft. Find its length and breadth.

37. The area of a rectangular field is 2000 sq. ft., and its length is 10 ft. more than its breadth. Find its dimensions.

38. After a man had lived in his house 4 months longer than the number of dollars monthly rental that he paid for his house, he had paid altogether $320 rent. How much was the monthly rental?

39. If d is the diagonal of a square of side s, then $d^2=2s^2$. Solve this equation for d. For s.

40. In Exercise 39, when $s=4$, find d, taking 1.414 as the square root of 2.

41. The volume (v) of a cylinder is the product of the area of the base (πr^2) and the height (h). That is, $v=\pi r^2h$. Solve this equation for r. For h.

42. Using $\frac{22}{7}$ for π in $v=\pi r^2h$, find the radius of the base of a cylinder, if its volume is $\frac{22}{112}$ sq. ft. and its altitude 4 ft.

43. Given $m=\dfrac{gv^2}{c^2}$. Express v in terms of the other letters.

44. In the preceding exercise, express c in terms of the other letters.

45. A room is 1 yd. longer than it is wide; at 75¢ per square yard, a covering for the floor of the room costs $31.50. Find the dimensions of the floor.

46. The length of a room is twice its height, and the breadth is 6 ft. more than the height. At 10¢ per square foot it costs $72 to decorate the side walls of the room, no allowance being made for openings. Find the dimensions of the room.

47. The cost of decorating a certain square ceiling is $45. If a second square ceiling of side 5 yd. longer were decorated at the same rate, the cost would be $80. Find the dimensions of the first ceiling.

48. A hall is lighted by a certain number of incandescent electric lights and 5 fewer of gas mantles. The candle power of each of the former is 70 greater than that of each mantle. The total candle power of the gas mantles is 500, and that of the electric lights 1800. How many lamps of each sort?

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define and illustrate a *complete* quadratic equation; also an *incomplete* quadratic equation. Sec. 369.

2. What is a *pure quadratic?* An *affected quadratic?* Sec. 370.

3. Explain the process of completing the square. Sec. 371.

4. Why approximate roots? Why not use such roots for testing? Sec. 372.

5. Explain how to proceed when the coefficient of x^2 is not unity. Sec. 373.

6. State the *formula* for solving quadratic equations. Sec. 373.

7. What kind of higher equations may be solved by quadratic methods? Sec. 374.

8. How are *radical equations* solved? Secs. 375, 377, 378.

9. Define *extraneous* roots. Sec. 376.

HISTORICAL NOTE

The solution of quadratic equations dates from Diophantos, but he did not leave any description of a general method. The Hindoos could solve special cases by completing the square when the number to be added for this purpose was obvious, and Cridharra is said to have formulated a rule for this. They discussed the existence of two roots, although they felt that positive roots only were significant, and Bhaskara called a negative root "inadequate, because people don't approve of negative roots." The Arab, Mohammed ben Musa, or Al-Khowarazmi (p. 93), helped to simplify the solution of equations by giving directions "first to transpose terms, then combine them," but added nothing else of importance.

The European scholars of the Middle Ages contented themselves with translating the Greek and the Arab manuscripts so that further development in solving equations awaited the mathematicians of the fifteenth century. By that time there were current more than twenty special rules for solving quadratics, which Michael Stifel (about 1550) reduced to three.

Stifel, the greatest German algebraist of the sixteenth century, was born in Esslingen and was educated by the monks for the ministry; but his interest in the mystic numbers found in the prophetic books of the Bible led him to study mathematics. The new science of algebra was called by the Germans "Coss," and Stifel is now known as the greatest "cossist." He made many improvements in algebra, but his limited idea of the negative number hindered him, particularly in the solution of equations. Thus, he succeeded only in decreasing the number of rules for solving the quadratic equation, whereas Stevin, in the next century, with full knowledge of the negative number, reduced all of these rules to our present method of completing the square.

CHAPTER XXIII

SYSTEMS OF QUADRATIC EQUATIONS

SIMULTANEOUS QUADRATIC EQUATIONS

380. Two simultaneous quadratic equations with two unknowns cannot in general be solved by the methods used in solving quadratic equations, because an equation of higher degree usually results from eliminating one of the unknowns. But many systems containing quadratic equations can be solved by quadratic methods, among them the following:

381. Class I. In which the result of substitution is a quadratic form.

1. *A system of equations composed of a linear equation and a quadratic equation can always be solved by substitution.*

EXAMPLE

Solve:
$$\begin{cases}3x+4y=24. & (1)\\ x^2+y^2=25. & (2)\end{cases}$$

From (1),
$$3x=24-4y. \qquad (3)$$

From (3),
$$x=8-\frac{4y}{3}. \qquad (4)$$

Substituting (4) in (2),
$$\left(8-\frac{4y}{3}\right)^2+y^2=25. \qquad (5)$$

Simplifying (5),
$$25y^2-192y+351=0. \qquad (6)$$

Factoring (6),
$$(y-3)(25y-117)=0. \qquad (7)$$

Solving (7),
$$y=3 \text{ and } y=\frac{117}{25}. \qquad (8)$$

From (4),
$$x=4 \text{ and } x=\frac{44}{25}. \qquad (9)$$

TEST.
$$\left.\begin{aligned}3\cdot 4+4\cdot 3=24\\ 3^2+4^2=25\end{aligned}\right\}\text{ and }\left\{\begin{aligned}&\frac{3\cdot 44}{25}+\frac{4\cdot 117}{25}=24.\\ &\left(\frac{44}{25}\right)^2+\left(\frac{117}{25}\right)^2=25.\end{aligned}\right.$$

2. *When both equations are quadratic, substitution is applicable if the result of substitution is an equation having the quadratic form.*

EXAMPLE

Solve: $$\begin{cases} x^2 + y^2 = 25, & (1) \\ xy = 12. & (2) \end{cases}$$

From (2), $x = \frac{12}{y}$. (3)

Substituting from (3) in (1), $\left(\frac{12}{y}\right)^2 + y^2 = 25$. (4)

Simplifying (4), we obtain an equation in quadratic form, $y^4 - 25\,y^2 + 144 = 0$. (5)

Factoring (5), $(y^2 + 16)(y^2 - 9) = 0$. (6)

Solving (6), $y = \pm 4$, and ± 3. (7)

Substituting (7) in (3), $x = \pm 3$, and ± 4. (8)

Test. Taking both values to be positive, or both to be negative,

$$(\pm 3)^2 + (\pm 4)^2 = 25.$$
$$(\pm 3) \cdot \pm 4 = 12.$$

There are frequently various methods of solving the same problem. Thus, in the last example, multiply (2) by 2, add it to or subtract it from (1); the resulting equations $(x + y)^2 = 49$, and $(x - y)^2 = 1$, can be solved by extracting the square roots of both members and adding and subtracting the resulting equations.

382. Corresponding Values. The sets of values of x and y which form solutions may be determined by noticing which value of one unknown furnishes a given value of the other in the process of solution.

Thus, in Example 1, p. 282, $y = 3$ produces $x = 4$.

WRITTEN EXERCISES

Solve and test:

1. $x^2 + y^2 = 2\,xy$, $x + y = 8$.
2. $x^2 + y^2 = 25$, $x + y = 1$.
3. $x^2 + y^2 = 13$, $2\,x + 3\,y = 13$.
4. $x - y = 1$, $x^2 - y^2 = 16$.
5. $m^2 - n^2 = 16$, $m - n = 2$.
6. $2\,R^2 - R_1{}^2 = 14$, $3\,R + R_1 = 11$.
7. $x + y = a$, $x^2 - y^2 = b$.
8. $x - y = b$, $xy = a^2$.
9. $2\,k = k^2 - k'^2$, $2\,k = 4\,kk'$.

383. Class II. In which one equation is homogeneous.

A system in which one equation has only terms of the second degree in x *and* y *can be solved by finding* x *in terms of* y, *or* vice versa, *from this equation and substituting in the other.*

EXAMPLE

Solve: $$\begin{cases} x^2 - 5xy + 6y^2 = 0, & (1) \\ x^2 - y^2 = 27. & (2) \end{cases}$$

Dividing (*1*) by y^2, $$\frac{x^2}{y^2} - \frac{5xy}{y^2} + \frac{6y^2}{y^2} = 0. \qquad (3)$$

Simplifying (*3*), $$\left(\frac{x}{y}\right)^2 - 5\left(\frac{x}{y}\right) + 6 = 0. \qquad (4)$$

Factoring (*4*), $$\left(\frac{x}{y} - 2\right)\left(\frac{x}{y} - 3\right) = 0. \qquad (5)$$

Solving (*5*), $$\frac{x}{y} = 2,\ \frac{x}{y} = 3. \qquad (6)$$

From (*6*), $$x = 2y,\ x = 3y. \qquad (7)$$

Substituting $x = 2y$ in (*2*), $$(2y)^2 - y^2 = 27. \qquad (8)$$

Solving (*8*), $$y = \pm 3. \qquad (9)$$

Substituting (*9*) in $x = 2y$, $$x = \pm 6. \qquad (10)$$

Substituting $x = 3y$ in (*2*), $$(3y)^2 - y^2 = 27. \qquad (11)$$

Solving (*11*), $$y = \frac{\pm 9}{2\sqrt{6}} = \pm \tfrac{3}{4}\sqrt{6}. \qquad (12)$$

Substituting (*12*) in $x = 3y$, $$x = \frac{\pm 27}{2\sqrt{6}} = \pm \tfrac{9}{4}\sqrt{6}. \qquad (13)$$

Test.

Taking both values to be positive or both negative, $$\begin{cases} (\pm 6)^2 - 5(\pm 3)(\pm 6) + 6(\pm 3)^2 = 0. \\ (\pm 6)^2 - (\pm 3)^2 = 27. \end{cases}$$

Taking the signs as before, $$\begin{cases} \left(\frac{\pm 27}{2\sqrt{6}}\right)^2 - 5\left(\frac{\pm 27}{2\sqrt{6}}\right)\left(\frac{\pm 9}{2\sqrt{6}}\right) + 6\left(\frac{\pm 9}{2\sqrt{6}}\right)^2 = 0. \\ \left(\frac{\pm 27}{2\sqrt{6}}\right)^2 - \left(\frac{\pm 9}{2\sqrt{6}}\right)^2 = 27. \end{cases}$$

Notes. 1. When the equation in $\frac{x}{y}$, as in step (4), cannot be factored by inspection, the formula for solving the quadratic equation is used.

2. When an equation has the right member zero, it is unnecessary to divide by x^2 or y^2, if the left member can be factored by inspection. Thus, in (*1*) above, $(x - 3y)(x - 2y) = 0$, hence $x = 3y$ and $x = 2y$, as in (*7*).

WRITTEN EXERCISES

Solve:

1. $x^2 - 2xy - 3y^2 = 0,$
$x^2 + 2y^2 = 12.$
2. $x^2 + xy - 2y^2 = 0,$
$x^2 + y^2 = 50.$
3. $x^2 - y^2 = 0,$
$x^2 - 3xy + 2y = \frac{1}{2}.$
4. $2x^2 - 5xy + 2y^2 = 0,$
$4x^2 - 4y^2 + 12x = 3.$
5. $m(m + n) = 0,$
$m^2 - mn + n^2 = 27.$
6. $6x^2 + 5xy + y^2 = 0,$
$y^2 - x - y = 32.$
7. $3t^2 - 2tu - u^2 = 0,$
$t + u + u^2 = 32.$
8. $x^2 + a^2y^2 = 0,$
$x^2 + ay - y^2 = a^2.$
9. $x^2 + 3x - 5y + xy = 400,$
$x^2 + 7xy + 10y^2 = 0.$
10. $4x^2 + 4xy + y^2 = 0,$
$x^2 + 3y^2 - 2x = 195.$

384. Class III. In which the unknowns are symmetrically involved.

A system in which each equation is unaltered when x *and* y *are interchanged can be solved by letting* $x = u + v$ *and* $y = u - v$.

EXAMPLE

Solve:
$$\begin{cases} x^2 + y^2 = 5, & (1) \\ xy + x + y = 5. & (2) \end{cases}$$

Let $x = u + v$, $y = u - v$,

then (1) becomes $(u + v)^2 + (u - v)^2 = 5,$ (3)

and (2) becomes

$(u + v)(u - v) + (u + v) + (u - v) = 5.$ (4)

Simplifying (3), $2u^2 + 2v^2 = 5.$ (5)

Simplifying (4), $u^2 - v^2 + 2u = 5.$ (6)

Dividing (5) by 2, and subtracting it from (6), $2u^2 + 2u = \frac{15}{2}.$ (7)

Simplifying, $u^2 + u - \frac{15}{4} = 0.$ (8)

Solving (8), $u = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 15},$ (9)

$= -\frac{1}{2} \pm 2 = 1\frac{1}{2}, -2\frac{1}{2}.$

Substituting in (5), $2 \cdot \frac{9}{4} + 2v^2 = 5.$ (10)

Solving (10), $v^2 = \frac{1}{4}$, and $v = -\frac{1}{2}, +\frac{1}{2}.$ (11)

Finally, $x = u + v = 1\frac{1}{2} \mp \frac{1}{2} = 1$ or $2,$ (12)

and, $y = u - v = 1\frac{1}{2} \pm \frac{1}{2} = 2$ or $1.$ (13)

If the other value, $u = -2\frac{1}{2}$, step (9), be substituted in (5), then $v = \pm \frac{1}{2}\sqrt{-15}$. But square roots of negative numbers, called imaginaries, will not be admitted until they have been studied in Chapter XXVIII.

Special systems of symmetric equations may be solved by dividing one equation by the other.

For example,

$$\begin{cases} x^3 + y^3 = 18, \\ x + y = 6, \end{cases} \quad \text{or} \quad \begin{cases} x^4 + x^2y^2 + y^4 = 8, \\ x^2 - xy + y^2 = 4. \end{cases}$$

WRITTEN EXERCISES

Solve by Sec. 384 or by some of the previous methods:

1. $x^2 - xy + y^2 - 7 = 0,$
   $x - y + 1 = 0.$
2. $x^2 - xy + y^2 = 12,$
   $x^2 + xy + y^2 = 4.$
3. $xy - (x + y) - 1 = 0,$
   $xy = -3.$
4. $x^2 + y^2 = 39,$
   $y - x = 3.$
5. $2x^2 + 2y^2 - (x - y) = 9,$
   $y^2 = 1.$
6. $x^2 - xy + y^2 = 19,$
   $xy = 15.$
7. $xy = 3(x + y),$
   $x^2 + y^2 = 160.$
8. $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{5}{2},$
   $\dfrac{1}{x} + \dfrac{1}{y} = 4.$
9. $x^2 + y^2 + x + y = 188,$
   $xy = 77.$
10. $x^3 - y^3 = 9,$
    $x - y = 3.$
11. $x^4 + x^2y^2 + y^4 = 8,$
    $x^2 + xy + y^2 = 2.$

385. The foregoing classes of simultaneous quadratic equations are applied in the following problems.

WRITTEN EXERCISES

1. Two square floors are paved with stones 1 ft. square; the length of the side of one floor is 12 ft. more than that of the other, and the number of stones in the two floors is 2120. Find the length of the side of each floor.

Suggestion. Let x be the length in feet of a side of the smaller floor and y be the length of the other, then

$$x = y - 12, \qquad (1)$$

and by the given conditions, $$x^2 + y^2 = 2120. \qquad (2)$$

Substituting (*1*) in (*2*), $$(y - 12)^2 + y^2 = 2120. \qquad (3)$$

Simplifying and factoring (*3*), $$(y - 38)(y + 26) = 0 \qquad (4)$$

The negative values not being admissible, the squares are 26 ft. and 38 ft. on a side.

2. The sum of the sides of two squares is 7 and the sum of their areas is 25. Find the side of each square.

3. The hypotenuse of a certain right triangle is 50, and the length of one of its sides is $\frac{3}{4}$ that of the other. Find the sides.

4. The difference between the hypotenuse of a right triangle and the other two sides is 3 and 6 respectively. Find the sides.

5. A number consists of two digits; the sum of their squares is 41. If each digit is multiplied by 5, the sum of these products is equal to the number. Find the number.

6. The difference between two numbers is 5; their product exceeds their sum by 13. Find the numbers.

7. The diagonal of a rectangle is 13 in.; the difference between its sides is 7 in. Find the sides.

8. The diagonal of a rectangle is 29 yd., and the sum of its sides is 41 yd. Find the sides.

9. The sum of the perimeters of two squares is 104 ft.; the sum of their areas is 346 sq. ft. Find their sides.

10. The difference between the areas of two squares is 231 sq. in.; the difference between their perimeters is 28 in. Find their sides.

11. Two trains leave New York simultaneously for St. Louis, which is 1170 mi. distant; the one goes 10 mi. per hour faster than the other and arrives $9\frac{3}{4}$ hr. sooner. Find the rate of each train.

12. In going 120 yd. the front wheel of a wagon makes 6 revolutions more than the rear wheel; but if the circumference of each wheel were increased 3 ft., the front wheel would make only 4 revolutions more than the rear wheel in going the same distance. Find the circumference of each wheel.

13. The sum of the volumes of two cubes is 35 cu. in. and the sum of the lengths of their edges is 5 in. Find the length of the edge of each.

14. The difference between the volumes of two cubes is 37 cu. ft. and the difference between their edges is 1 ft. Find their edges.

386. Higher Equations. Certain systems containing higher equations can be solved by the methods of this chapter.

EXAMPLES

1. Solve:
$$\begin{cases} x^3+y^3=18\,xy, & (1)\\ x+y=12. & (2)\end{cases}$$

Put $x=u+v$, and $y=u-v$,

Then, from (*1*) $(u+v)^3+(u-v)^3=18(u+v)(u-v)$, (*3*)

and from (*2*), $(u+v)+(u-v)=2\,u=12.$ (*4*)

$\therefore u=6.$

Combining (*4*) and (*3*), $216+18\,v^2=9(36-v^2).$ (*5*)

Solving (*5*), $v=\pm 2.$ (*6*)

From (*4*) and (*6*), $x=u+v=8$ and 4, (*7*)

and $y=u-v=4$ and 8. (*8*)

Test as usual.

2. Solve:
$$\begin{cases} x^4+y^4=706, & (1)\\ x-y=2. & (2)\end{cases}$$

Put $x=u+v$, $y=u-v$,

Then from (*1*), $(u+v)^4+(u-v)^4=706,$ (*3*)

and from (*2*), $(u+v)-(u-v)=2.$ (*4*)

Simplifying (*4*), $v=1.$ (*5*)

From (*5*) and (*3*), $(u+1)^4+(u-1)^4=706.$ (*6*)

Simplifying (*6*), $u^4+6\,u^2-352=0.$ (*7*)

Solving (*7*), $u^2=16,$ (*8*)

and $u^2=-22.$ (*9*)

From (*8*), $u=\pm 4.$ (*10*)

From (*5*) and (*10*), $x=\pm 4+1=5,\ -3.$ (*11*)

and $y=\pm 4-1=-5,\ +3.$ (*12*)

The imaginary values of u in (9), namely $\pm\sqrt{-22}$, would give other values of x and y, but such values will be omitted here and considered in Chapter XXVIII.

Test as usual.

WRITTEN EXERCISES

Solve:

1. $x^3+y^3=189,$
   $x+y=9.$
2. $x^3+y^3=72,$
   $x+y=6.$
3. $x^3+y^3=189,$
   $x^2y+xy^2=180.$
4. $x^4+y^4=2,$
   $x+y=0.$
5. $x^5+y^5=2,$
   $x+y=2.$
6. $x^4+y^4=17,$
   $x-y=1.$

REVIEW

WRITTEN EXERCISES

Solve:

1. $x + y = 7.5,$
$xy = 14.$

2. $3x - 2y = 0,$
$xy = 13.5.$

3. $x + y = 7,$
$x^2 - y^2 = 21.$

4. $x - y = 5,$
$x^2 + y^2 = 37.$

5. $x - y = 1,$
$3x^2 + y^2 = 31.$

6. $x - y = 5,$
$x^2 + 2xy + y^2 = 75.$

7. $x + y = 7(x - y),$
$x^2 + y^2 = 225.$

8. $5(x^2 - y^2) = 4(x^2 + y^2),$
$x + y = 8.$

9. $\dfrac{x^2y}{x + 5y} = 4,$
$xy = 20.$

10. $x^2 + xy + y^2 = 19,$
$xy = 6.$

11. $(x + 2)(x - 3) = 0,$
$x^2 + 3xy + y^2 = 5.$

12. $x + y = 5,$
$\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6}.$

13. $x - y = -3,$
$x^3 - y^3 = -9.$

14. $x^2 - y^2 = 144,$
$x - y = 8.$

15. $\dfrac{x}{2} - \dfrac{y}{3} = 0,$
$x^2 + y^2 = 5(x + y) + 2.$

16. $x + y = 9,$
$\dfrac{xy}{\sqrt{xy}} = \dfrac{10}{\sqrt{5}}.$

17. $x^3 - y^3 = 665,$
$x - y = 5.$

18. $x + y = \dfrac{x}{y} = x^2 - y^2.$

19. $x + y = a,$
$x^3 + y^3 = b^3.$

20. $x - y = a,$
$x^3 - y^3 = b^3.$

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Name three classes of *simultaneous quadratic equations* that can be solved by quadratic methods. Secs. 381, 383, 384.

2. State how the first class may be solved; also the second class; the third class. Secs. 381, 383, 384.

3. How is it possible to determine what values of one unknown are to be associated with those of the other? Sec. 382.

4. State a set of higher equations that can be solved by quadratic methods. Sec. 386.

HISTORICAL NOTE

Simultaneous quadratic equations in determinate systems, such as we study in elementary algebra, received little attention until recent times. Mathematicians of the past favored the study of indeterminate systems, that is, systems in which there are more unknown quantities than equations. These problems offered a wide range for ingenuity, and Diophantos invented many special devices for their solution. The Hindoos and Arabs were likewise attracted to this class of problems. Thus, no real progress was made by Eastern scholars in finding general solutions of simultaneous quadratic equations.

To illustrate how little was known about this subject in the middle ages, we may refer to Gerbert (990) who, while teaching at Rheims, gained much fame as a mathematician by solving the problem: *To find the sides of a right-angled triangle given its area and hypotenuse.* It seems strange, from our present day point of view, that the solution of so simple a pair of equations as:

$$(1)\quad x^2 + y^2 = h^2 \qquad\qquad (2)\quad \tfrac{1}{2}xy = a$$

should have brought great distinction to any one. Gerbert is often referred to as Sylvester, because he later became Pope under the name of Sylvester II.

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